MMYovanovich
890808

HTD-Vol. 107

# HEAT TRANSFER IN CONVECTIVE FLOWS

*presented at*
THE 1989 NATIONAL HEAT TRANSFER CONFERENCE
PHILADELPHIA, PENNSYLVANIA
AUGUST 6–9, 1989

*sponsored by*
THE HEAT TRANSFER DIVISION

*edited by*
R. K. SHAH
GENERAL MOTORS CORPORATION

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
United Engineering Center 345 East 47th Street New York, N.Y. 10017

1989 National Heat Transfer Conference
HTD-Vol. 107, Heat Transfer in Convective Flows

# NATURAL CONVECTION FROM A VERTICAL PLATE WITH STEP CHANGES IN SURFACE HEAT FLUX

**S. Lee and M. M. Yovanovich**
Microelectronics Heat Transfer Laboratory
Department of Mechanical Engineering
University of Waterloo
Waterloo, Ontario, Canada

## ABSTRACT

An approximate analytical model has been developed to predict heat transfer and flow characteristics of a steady state, two dimensional laminar boundary layer in the vicinity of a vertical flat plate under natural convection. The plate dissipates heat, with multi-step changes in surface heat flux of an arbitrary pattern, into an extensive, stagnant fluid which is maintained at uniform temperature. Wall temperature and maximum velocity variations are predicted and compared with existing numerical data, which were obtained by solving the boundary layer equations for cases with a number of uniform heat flux sources mounted flush on a vertical adiabatic plate in air using finite difference methods. The agreement is good. In order to examine the validity and accuracy of the present model, additional test computations were carried out for situations in which either the exact solutions or the proper behavior of the solutions are known. The model is shown to be valid and discussions are included herein as a result.

## NOMENCLATURE

| | |
|---|---|
| $\tilde{C}$ | parameter, $\tilde{C} = (\sqrt{\Pr}\,\tilde{C}_u/4)^{2/3}$ or $\tilde{C} = \Pr/(4\tilde{C}_\eta^2)$ |
| $\tilde{C}_u, \tilde{C}_\eta$ | parameters defined in Eqs. (18), and (19) or (20) |
| $\operatorname{erfc}\eta$ | Complementary Error function, $\frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-\tau^2}\,d\tau$ |
| $f_T, f_u$ | functions defined by Eqs. (11) and (12) |
| $g$ | gravitational acceleration |
| $G_e, G_m$ | functions given in Table 1 |
| $\tilde{h}_e, \tilde{h}_m$ | functions given in Table 2 |
| $\mathcal{H}_e, \mathcal{H}_m$ | functions expressed by Eqs. (45) and (44) |
| $\mathrm{i}^n\operatorname{erfc}\eta$ | Complementary Error function integrated $n$-times, $\int_\eta^\infty \mathrm{i}^{n-1}\operatorname{erfc}\tau\,d\tau$, $\mathrm{i}^1\operatorname{erfc}\eta = \operatorname{ierfc}\eta$, $\mathrm{i}^0\operatorname{erfc}\eta = \operatorname{erfc}\eta$ |
| $k$ | thermal conductivity of fluid |
| $p$ | square root of Prandtl number, $\sqrt{\Pr}$ |
| $q$ | local heat-flux |
| $q^*$ | dimensionless local heat-flux, $q/q_{w_0}$ |
| $q_e^*, q_m^*$ | expressions given by Eqs. (41) and (40) |
| $t$ | time variable defined in $t-y$ plane |
| $T$ | temperature excess over ambient fluid temperature |
| $u$ | local velocity parallel to plate, in $x$-direction |
| $u_c$ | characteristic velocity across the boundary layer |
| $U$ | dimensionless $u$-velocity defined by Eq. (47) |
| $v$ | local velocity normal to plate, in $y$-direction |
| $x$ | vertical coordinate measured from leading edge |
| $y$ | horizontal coordinate measured from plate surface |

**Greek Symbols**

| | |
|---|---|
| $\alpha$ | thermal diffusivity of fluid, $k/(\rho c_p)$ |
| $\beta$ | thermal expansion coefficient, $-(\partial\rho/\partial T)_P/\rho$ |
| $\gamma$ | function defined by Eq. (38) |
| $\eta_0$ | variable defined by Eq. (13) or Eq. (48) |
| $\eta_i$ | variable defined by Eq. (14) or Eq. (48) |
| $\nu$ | kinematic viscosity, $\mu/\rho$ |
| $\xi$ | dimensionless $x$-coordinate, $x/x_0$ |
| $\tilde{\phi}, \tilde{\varphi}, \tilde{\psi}$ | modifying functions |
| $\tilde{\Phi}$ | function defined by Eq. (39) |
| $\theta$ | dimensionless temperature defined by Eq. (46) |

**Subscripts**

| | |
|---|---|
| 0 | parameters at leading section |
| $i$ | parameters at $i$-th step |
| $r$ | references |
| $w$ | wall conditions |

**Dimensionless Groups**

| | |
|---|---|
| $G^*$ | modified Grashof number, $g\beta q_r x_r^4/k\nu^2$ |
| $Gr_x$ | Grashof number, $g\beta T_{w_0} x^3/\nu^2$ |
| $Gr_x^*$ | modified Grashof number, $g\beta q_{w_0} x^4/k\nu^2$ |
| $Nu_x$ | Nusselt number, $q_{w_0} x/T_{w_0} k$ |
| Pr | Prandtl number, $\nu/\alpha$ |

## INTRODUCTION

The natural convection heat transfer from a vertical flat plate has been a subject of numerous investigations in the past few decades. The plate with thermal conditions that allow similarity transformations have been examined by Ostrach (1953), Sparrow and Gregg (1956, 1958), and Jaluria and Gebhart (1977). They have considered steady state, two dimensional laminar boundary layer equations for uniform wall temperature, uniform surface heat flux, excess wall temperature variations of the power and exponential forms, and a line source on an adiabatic plate. Yang (1960) found that there are no other types of boundary conditions which would make similarity solutions possible for steady natural convection from a vertical plate. Numerous studies have been carried out to expand available solutions for non-similar boundary conditions by using various methods. Unfortunately, analytical solution techniques for problems with arbitrary boundary conditions, that are frequently expected from most practical applications, are not available.

The modeling of an isolated vertical flat plate with arbitrary surface thermal conditions would be useful in many technological applications such as the thermal design of Printed Circuit Boards (PCBs) on which a number of finite sized heat sources are mounted. Complete thermal phenomena involved in the final product of a PCB are too complex and are impractical to analyze as a whole. A PCB is often modeled as a flat plate in thermal analyses with heat sources mounted flush with its surface. Bar-Cohen (1985) has discussed and found that it is possible to apply heat transfer relations developed for a smooth wall to non-smooth component carrying PCBs. Nonetheless, the sources are discrete and randomly distributed in general. The heat transfer associated with PCB applications is usually a conjugate heat transfer, in which all three modes of heat transfer, namely heat conduction through the board, heat convection in the ambient fluid and surface radiation to the surroundings, may occur simultaneously. It becomes apparent that neither the resulting wall temperature nor convective heat flux variations would be known *a priori*, and similarity transformations would rarely be allowed in the analyses.

During the early stages of an ongoing development towards conjugate thermal modeling of a PCB cooled by natural convection, the present authors adopted the flat plate approximation but recognized a need for a model that can predict wall temperature variations in a vertical plate when arbitrary surface heat flux variations are prescribed. Only then, by means of an iterative procedure, would the model become capable of simulating conjugate heat transfer which is ubiquitous in situations involving PCBs (Lee and Yovanovich, 1989). As previously mentioned, an analytical model capable of dealing with problems under the present consideration does not exist in the literature. Other techniques to obtain solutions for cases with the arbitrary thermal conditions may include experimental investigations and fully numerical methods. The data from experiments are, however, those of case-by-case studies and cannot be manipulated to predict results of other cases, since their range of reliable application is mostly limited within the range of parameters that are examined. Utilization of fully numerical methods such as finite difference or finite element methods usually requires a main-frame computer with associated high computing expenses, and may become impractical in some situations. Many applications call for simpler and inexpensive solution techniques, though they may be less accurate than more rigorous and time consuming methods (Schetz, 1963).

In this paper, an approximate analytical model is presented which can be used to predict two dimensional laminar natural convection flow and heat transfer about a vertical flat plate dissipating energy into a quiescent medium with multi-step changes in surface heat flux of an arbitrary pattern. With sufficient discretization, the model can be applied to most surface heat flux variations that can be expected from a flat plate model of PCBs.

Other approximate methods are available in the literature for cases with a continuous variation in the thermal boundary conditions (Tribus, 1958, Raithby and Hollands, 1975). However, applicability of these results is limited to problems that closely maintain the form of the specified polynomials in characterizing the temperature and velocity profiles across the boundary layer. This deficiency in Tribus's solutions was pointed out and further discussed by Sparrow and Gregg in their "Authors Closure" following Tribus's discussion. In addition, Zinnes (1970) has shown that, when the surface of a plate experiences abrupt thermal variations, the results obtained by using the model of Tribus considerably deviate from those obtained by using finite difference methods.

The downstream wall temperature and maximum velocity variations are obtained by using the present model and compared with existing numerical data of Jaluria (1982), who solved the boundary layer equations by using finite difference methods for cases with a number of strip thermal sources of uniform surface heat flux mounted flush on an adiabatic plate in air. The comparison resulted in satisfactory to good agreement. Since the present model is developed based on an approximate method, computations are carried out to examine the validity and accuracy of the model. A complete verification over all ranges of parameters is obviously neither practical nor necessary for the present purpose. The validity of the model is, therefore, demonstrated within the range of parameters for comparisons with the results of the aforementioned numerical data. The model is further validated by examining its behavior using the cases for which either the exact solutions or the proper behavior of the solutions are known.

## PROBLEM STATEMENT

The geometric configuration and coordinate system of the problem are depicted in Fig. 1, where a vertical flat plate is shown with a possible step variation in surface heat flux. The problem is two dimensional in the $x-y$ plane, having a large transverse dimension in the $z$-direction normal to the page. The plate is dissipating heat into an extensive, quiescent fluid which is assumed to be maintained at uniform temperature. As shown in the figure, the value of the sectional heat flux, except $q_{w_0}$ in the leading section, may be zero; the plate is insulated, or negative; the fluid is heating the plate, so long as the resulting temperature excess over the ambient fluid temperature at any point within the boundary layer is maintained positive.

A set of boundary layer equations that governs two dimensional, steady state momentum and energy transport in natural convection is written below.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta T \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The boundary conditions associated with the above equations are

$$\begin{aligned}
&\text{at } y = 0\,, && u = v = 0, \\
& && -k\frac{\partial T}{\partial y} = q_{w_0} \ \text{ for } 0 < x \le x_0 \\
& && -k\frac{\partial T}{\partial y} = q_{w_i} \ \text{ for } x_{i-1} < x \le x_i \ ; \ i = 1,\, 2,\, 3,\, \ldots \\
&\text{as } y \to \infty\,, && u \to 0, \ T \to 0 \\
&\text{at } x = 0\,, && u = T = 0
\end{aligned} \tag{4}$$

where $x$ and $y$ are the coordinates parallel and normal to the plate, $u$ and $v$ are the corresponding components of the velocity, and $T$ is the local temperature excess over the ambient fluid temperature. All $q_{w_i}$ including $q_{w_0}$ are uniform. The usual assumptions and approximations, such as those of constant fluid properties except the density in the derivation of the buoyant term and negligible viscous heating, are made in deriving these equations. The present problem includes the cases with a step change in surface heat flux when $i = 1$, which were inclusively examined by the present authors in an earlier study (Lee and Yovanovich, 1988).

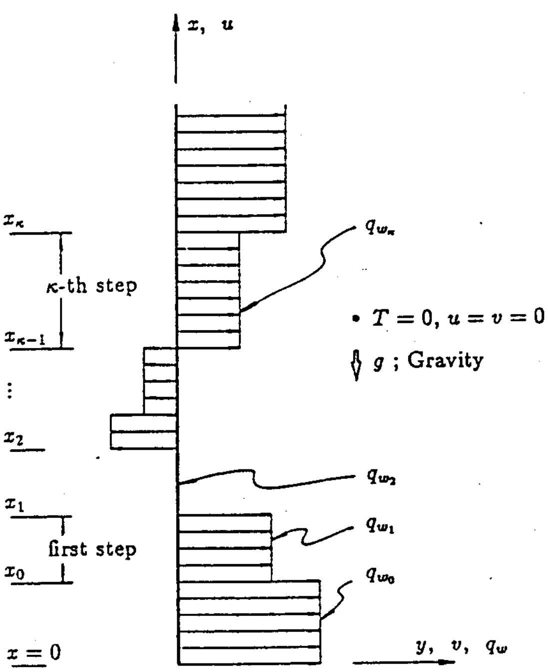


Figure 1: Geometric configuration shown with a schematic surface heat flux variation.

## Analysis

An exact analytical solution to the above problem does not exist, and obtaining such solutions in the near future seems improbable. As was done in the earlier study, the non-linear convection terms in the left hand side of the momentum and energy equations, Eqs. (2) and (3), are linearized through two stages as follows.

Firstly, a transient coordinate $t$ is introduced through a $t - x$ transformation defined as

$$t = \frac{x}{u_c} \tag{5}$$

where $u_c$ is called the characteristic velocity. By the above definition, it can be said that the distance, measured from the leading edge of the plate at $x = 0$, to a location along the $x$-coordinate corresponds to a specific lapse in time. The original $x$-coordinate is hence transformed into a transient coordinate specified by the time variable $t$. The characteristic velocity is presently an unknown function of $x$. It may be viewed as an effective mean flow velocity of the fluid in the boundary layer initiated at the leading edge of the plate.

Secondly, upon this transformation, an assumption is made such that diffusion is dominant in transporting both heat and momentum across the boundary layer in the $y$-direction at fixed time $t$. The effect of convection transport in the $t - y$ plane is thereby neglected, and profiles of the temperature and velocity distributions in the original $x - y$ plane will be determined approximately by transient diffusion equations in the $t - y$ plane. Subsequently, the above governing differential equations are linearized in the $t - y$ plane and the transformed momentum and energy equations take forms of

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta T \tag{6}$$

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{7}$$

respectively. This set of equations is identical to the governing differential equations which describe *real time* transient natural convection heat transfer from an infinite plate. The continuity equation, Eq. (1), is obsolete since the $u$-velocity becomes one dimensional in $y$ at fixed time $t$. The above assumption of dominant diffusion in the $y$-direction in the $t - y$ plane can only be validated indirectly through comparisons of the resulting temperature and velocity distributions with existing and known data, as will be carried out in the following section. The corresponding transient boundary conditions that are compatible with those given by Eq. (4) can now be given as

$$\begin{aligned}
&\text{at } y = 0\,, && u = 0\,, \\
& && -k\frac{\partial T}{\partial y} = q_{w_0} \ \text{ for } 0 < t \le t_0 \\
& && -k\frac{\partial T}{\partial y} = q_{w_i} \ \text{ for } t_{i-1} < t \le t_i \ ; \ i = 1,\, 2,\, 3,\, \ldots \\
&\text{as } y \to \infty\,, && u \to 0, \ T \to 0 \\
&\text{at } t = 0\,, && u = T = 0
\end{aligned} \tag{8}$$

where $t_0$ and $t_i$ are the times corresponding to the locations at $x_0$ and $x_i$, respectively.

The transient solutions to the above equations are found for the temperature and velocity distributions by means of similarity transformations and the method of superposition (Lee, 1988). They are, for the $\kappa$-th step defined for $t_{\kappa-1} < t \le t_\kappa$,

$$T = \frac{2\sqrt{\alpha}}{k}\left[q_{w_0} t^{1/2} f_T(\eta_0) + \sum_{i=1}^{\kappa}(q_{w_i} - q_{w_{i-1}})(t - t_{i-1})^{1/2} f_T(\eta_i)\right] \quad (9)$$

$$u = \frac{4\sqrt{\alpha} g\beta}{k}\left[q_{w_0} t^{3/2} f_u(\eta_0) + \sum_{i=1}^{\kappa}(q_{w_i} - q_{w_{i-1}})(t - t_{i-1})^{3/2} f_u(\eta_i)\right] \quad (10)$$

where

$$f_T(\eta) = \mathrm{ierfc}\,\eta \quad (11)$$

$$f_u(\eta) = \begin{cases} \eta\, \mathrm{i}^2\mathrm{erfc}\,\eta & \text{for } \mathrm{Pr} = 1 \\ \dfrac{2}{1-\mathrm{Pr}}\left(\mathrm{i}^3\mathrm{erfc}\,\eta - \mathrm{i}^3\mathrm{erfc}\,\dfrac{\eta}{\sqrt{\mathrm{Pr}}}\right) & \text{for } \mathrm{Pr} \ne 1 \end{cases} \quad (12)$$

$$\eta_0 = \frac{y}{2\sqrt{\alpha t}} \quad (13)$$

$$\eta_i = \frac{y}{2\sqrt{\alpha(t - t_{i-1})}} \quad (14)$$

These solutions are exact to the set of transient equations, Eqs. (6) to (8), which approximately represents the set of original steady state equations, Eqs. (1) to (4), through the $t - x$ transformation. Determination of $t$, as well as $t - t_{i-1}$, in terms of $x$ would, therefore, render these equations to become approximate solutions to the original problem in the $x - y$ plane. Although the present analysis is based on the linearization of the equations in the $t - y$ plane, it is to be emphasized that the non-linearity of the steady state problem in the $x - y$ plane will be restored by treating each and every $t - x$ transformation, that is associated with the time lapse corresponding to the displacement within each step, to be specific as detailed below.

Consider the case of uniform surface heat flux, or, equivalently, the leading section of the present problem between $x = 0$ and $x = x_0$. Owing to the boundary layer approximations, solutions at the upstream locations are unaffected by the thermal variations imposed at the downstream locations. The solutions in this section, where $\kappa = 0$, are thus given solely by the first terms in Eqs. (9) and (10), in which $t$ is the only unknown parameter in terms of the steady state variables.

It was found in the earlier study (Lee and Yovanovich, 1988) that the required $t - x$ transformation for $0 < x \le x_0$ is

$$t = \frac{x}{u_c} = \tilde{C}\,\frac{x}{\frac{\nu}{x}\,\mathrm{Gr}_x^{*\,2/5}} \quad (15)$$

where $\tilde{C}$ is a function only of the Prandtl number, Pr, and is given by two different expressions. They are

$$\tilde{C} = \frac{\mathrm{Pr}}{4\tilde{C}_\eta^2} \quad (16)$$

and

$$\tilde{C} = \left(\frac{\sqrt{\mathrm{Pr}}\,\tilde{C}_u}{4}\right)^{2/3} \quad (17)$$

where $\tilde{C}_u$ and $\tilde{C}_\eta$ were determined by using the integral method.

$$\tilde{C}_u^{-5} = \frac{\mathrm{Pr}}{\pi^{3/2}} G_e \left(\frac{7G_m}{5} + \frac{2G_e}{1 + 1/\sqrt{\mathrm{Pr}}}\right)^2 \quad (18)$$

Table 1 : $G_e$ and $G_m$

| | $\mathrm{Pr} = 1$ | $\mathrm{Pr} \ne 1 \ (p = \sqrt{\mathrm{Pr}})$ |
|---|---|---|
| $G_e$ | $\frac{7 - 4\sqrt{2}}{120}$ | $\frac{2p^5 + (8\sqrt{2} - 9)p^3 + 5p^2 + 2 - 2(p^2+1)^{5/2}}{60\,p^3(1-p^2)}$ |
| $G_m$ | $\frac{19 - 13\sqrt{2}}{105}$ | $\frac{2p^7 + 7p^5 + (8\sqrt{2} - 9)(p^4 + p^3) + 7p^2 + 2 - 2(p^2+1)^{7/2}}{52.5\,p^3(1-p^2)^2}$ |

$$\tilde{C}_\eta^2 = \frac{\mathrm{Pr}}{\pi^{1/2}} G_e \tilde{C}_u \quad (19)$$

where $G_e$ and $G_m$ are tabulated in Table 1.

The two different expressions, as given by Eqs. (16) and (17), for $\tilde{C}$ are necessary in order to conserve concurrently the energy and momentum flows in the boundary layer. The $t - x$ transformation, Eq. (15), with $\tilde{C}$ given by Eq. (16) transforms $t$ in the similarity variable $\eta_0$ and $t^{1/2}$ in the temperature solution, Eq. (9). Equation (15) with $\tilde{C}$ given by Eq. (17) transforms $t^{3/2}$ occurring in the velocity solution, Eq. (10).

A value of $\tilde{C}_\eta$ may alternatively be found based on any natural convection heat transfer results obtained for a vertical plate with uniform surface heat flux. For example, using the correlation equation presented by Fujii and Fujii (1976) for $\mathrm{Nu}_x$ as a function of Pr and $\mathrm{Gr}_x^*$, $\tilde{C}_\eta$ can be expressed as

$$\tilde{C}_\eta = \frac{1}{\sqrt{\pi}}\frac{\mathrm{Nu}_x}{\mathrm{Gr}_x^{*\,1/5}} = \frac{1}{\sqrt{\pi}}\left(\frac{\mathrm{Pr}^2}{4 + 9\sqrt{\mathrm{Pr}} + 10\,\mathrm{Pr}}\right)^{1/5} \quad (20)$$

For the transformation of the time variables in the succeeding step sections, the $t - x$ transformation defined by Eq. (5) can further be generalized for the time lapse $t - t_{i-1}$ associated within the $i$-th step as

$$t - t_{i-1} = \frac{x - x_{i-1}}{u_c} \qquad \text{for } i = 1,\ 2,\ 3,\ \ldots \quad (21)$$

Upon this transformation, the problem is now reduced to determining the characteristic velocities introduced at every section of the discretized domain. For the $\kappa$-th step change, introduced beyond $x = x_{\kappa-1}$, all the existing characteristic velocities, that are determined and used to define the $t - x$ transformations up to the $(\kappa - 1)$-th step, have to be modified. Furthermore, an additional characteristic velocity has to be determined which would represent the flow within the new boundary layer evolved from the surface at the beginning of the $\kappa$-th step. The number of unknown modifying functions required to determine the characteristic velocities, and in turn, complete the solutions in the $\kappa$-th step, becomes $\kappa + 1$. This number can be seen also from the above transient solutions, Eqs. (9) and (10), as the total number of functions required to transform $t$ and $t - t_{i-1}$, for $i$ from 1 to $\kappa$, is $\kappa + 1$.

In order to clarify this, consider a problem with two step changes in surface heat flux. The $t - x$ transformations within the first step, $x_0 < x \leq x_1$, are defined as

$$t = \tilde{C}\,\tilde{\phi}_1 \frac{x}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{22}$$

$$t - t_0 = \tilde{C}\,\tilde{\psi}_1 \frac{x - x_0}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{23}$$

A subscript 1 is used to denote $\tilde{\phi}_1$ and $\tilde{\psi}_1$ as the modifying functions incurred by the first step and $\tilde{C}$ is given previously. $\tilde{\phi}_1$ is the function that modifies the primary characteristic velocity established from the leading edge of the plate, whereas $\tilde{\psi}_1$ is the function required in determining the characteristic velocity within the secondary boundary layer evolved at the beginning of the first step, $x = x_0$. They were determined by using the integral method in the earlier study (Lee and Yovanovich, 1988).

As soon as the second step is introduced, another boundary layer evolves from the surface at the beginning of the second step. This additional boundary layer grows within the existing secondary boundary layer in much the same way as the secondary boundary layer itself evolved when the first step was introduced. Not only do the two existing characteristic velocities as found in the above equations, namely $\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}/\tilde{C}\tilde{\phi}_1$ and $\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}/\tilde{C}\tilde{\psi}_1$, have to be modified, but a new characteristic velocity for the flow in the additional boundary layer has to be determined also. Therefore, two additional functions, say $\tilde{\phi}_2$ and $\tilde{\varphi}_2$, are required to modify the two existing characteristic velocities, and a third function, $\tilde{\psi}_2$, is introduced to determine the characteristic velocity within the new boundary layer. The resulting transformation functions that are associated with the second step, $x_1 < x \leq x_2$, may then be written as

$$t = \tilde{C}\,\tilde{\phi}_1\tilde{\phi}_2 \frac{x}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{24}$$

$$t - t_0 = \tilde{C}\,\tilde{\psi}_1\tilde{\varphi}_2 \frac{x - x_0}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{25}$$

$$t - t_1 = \tilde{C}\,\tilde{\psi}_2 \frac{x - x_1}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{26}$$

The above set of transformation functions show that there are three unknown modifying functions in all, due to the introduction of the second step.

It was found (Lee, 1988) that the heat transfer characteristic of the flow within the first step is insensitive to induced changes in the $\tilde{\phi}_1$ variations. This observation may be postulated to yield a general interpretation, and as such, the resulting heat transfer phenomena are not strongly dependent on those modifying functions that modify *existing* characteristic velocities. Moreover, the $\tilde{\phi}_1$ variations are shown to respond gradually to the step changes at the immediate downstream locations beyond $x = x_0$, in the region where the greatest portion of the changes in heat transfer rate take place. The abrupt changes in the heat transfer rate, observed right after the step, are mostly attributed to the changes in $\tilde{\psi}_1$ variations. Further modifying functions, $\tilde{\phi}_2$ and $\tilde{\varphi}_2$, that are introduced to modify *existing* characteristic velocities due to the addition of the second step, are expected to behave not only qualitatively similar to $\tilde{\phi}_1$, but also quantitatively similar to each other. In all, they are assumed to be unique, $\tilde{\varphi}_2 = \tilde{\phi}_2$, and hence, the above set of transformations become

$$t = \tilde{C}\,\tilde{\phi}_1\tilde{\phi}_2 \frac{x}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{27}$$

$$t - t_0 = \tilde{C}\,\tilde{\psi}_1\tilde{\phi}_2 \frac{x - x_0}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{28}$$

$$t - t_1 = \tilde{C}\,\tilde{\psi}_2 \frac{x - x_1}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{29}$$

Two functions, $\tilde{\phi}_2$ and $\tilde{\psi}_2$, are now required to be determined.

Although the approximation leading to the reduction of the number of unknown modifying functions to two was based on the observation that primarily concerns temperature distributions, the same may be applied for the velocity distributions. Similar approximations are introduced as additional step changes are considered. As a result, $(\kappa - 1)$ $t - x$ transformations, required due to the induction of the $\kappa$-th step between $x = x_{\kappa-1}$ and $x = x_\kappa$, are defined as

$$t = \tilde{C}\,\tilde{\phi}_1\tilde{\phi}_2\tilde{\phi}_3\cdots\tilde{\phi}_\kappa \frac{x}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{30}$$

$$t - t_0 = \tilde{C}\,\tilde{\psi}_1\tilde{\phi}_2\tilde{\phi}_3\cdots\tilde{\phi}_\kappa \frac{x - x_0}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{31}$$

$$t - t_1 = \tilde{C}\,\tilde{\psi}_2\tilde{\phi}_3\cdots\tilde{\phi}_\kappa \frac{x - x_1}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{32}$$

$$\vdots$$

$$t - t_{i-1} = \tilde{C}\,\tilde{\psi}_i\tilde{\phi}_{i+1}\cdots\tilde{\phi}_\kappa \frac{x - x_{i-1}}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{33}$$

$$\vdots$$

$$t - t_{\kappa-1} = \tilde{C}\,\tilde{\psi}_\kappa \frac{x - x_{\kappa-1}}{\frac{\nu}{x}\mathrm{Gr}_x^{*2/5}} \tag{34}$$

Each additional $\kappa$-th step requires the determination of two additional modifying functions, $\tilde{\phi}_\kappa$ and $\tilde{\psi}_\kappa$, that are dependent on the history of the surface thermal condition up to and including the step. These functions are determined, again, by applying the integral method. After manipulating, simplifying and rearranging expressions, the integral method results in a set of equations as

$$\frac{d\gamma_\kappa^3}{d\xi} = \frac{\frac{3}{5\xi q_e^*}\left(q_m^*\frac{\mathcal{H}_{e_\kappa}}{\mathcal{H}_{m_\kappa}} - q_{w_\kappa}^*\right) - \sum_{i=1}^{\kappa-1}\left[\left(\frac{\partial\mathcal{H}_{m_\kappa}/\partial\gamma_i}{7\gamma_i^2\mathcal{H}_{m_\kappa}} - \frac{\partial\mathcal{H}_{e_\kappa}/\partial\gamma_i}{5\gamma_i^2\mathcal{H}_{e_\kappa}}\right)\frac{d\gamma_i^3}{d\xi}\right]}{\frac{\partial\mathcal{H}_{m_\kappa}/\partial\gamma_\kappa}{7\gamma_\kappa^2\mathcal{H}_{m_\kappa}} - \frac{\partial\mathcal{H}_{e_\kappa}/\partial\gamma_\kappa}{5\gamma_\kappa^2\mathcal{H}_{e_\kappa}}} \tag{35}$$

$$\tilde{\Phi}_\kappa = \left(\frac{q_e^*}{\mathcal{H}_{e_\kappa}}\right)^{2/5} \tag{36}$$

where the initial condition for $\gamma_\kappa^3$ is

$$\lim_{\xi\to\xi_{\kappa-1}^+}\gamma_\kappa^3 = 0 \tag{37}$$

and

$$\gamma_i = \sqrt{\frac{\tilde{\psi}_i}{\tilde{\Phi}_i}\left(1 - \frac{\xi_{i-1}}{\xi}\right)} \tag{38}$$

$$\tilde{\Phi}_i = \prod_{j=1}^{i}\tilde{\phi}_j \tag{39}$$

$$q_m^* = 1 + \sum_{i=1}^{\kappa}(q_{w_i}^* - q_{w_{i-1}}^*)\gamma_i^2 \tag{40}$$

Table 2 : $\tilde{h}_m(\gamma)$ and $\tilde{h}_e(\gamma)$

| | | |
|---|---|---|
| $\mathrm{Pr}=1$ | $\tilde{h}_m(\gamma)$ | $\dfrac{12\gamma^7+7\gamma^5+7\gamma^2+12+35\gamma^2(\gamma^2+1)^{3/2}-12(\gamma^2+1)^{7/2}}{19-13\sqrt{2}}$ |
| | $\tilde{h}_e(\gamma)$ | $\dfrac{2\gamma^5+5\gamma^3+5\gamma^2+2-2(\gamma^2+1)^{5/2}}{7-4\sqrt{2}}$ |
| $\mathrm{Pr}\neq 1$ | $\tilde{h}_m(\gamma)$ | $\left[(p^2-1)(p-1)\left\{7p^2\gamma^2(\gamma^3+1)+2(p^2+1)(p^2+p+1)(\gamma^7+1)\right\} +2p^3(p+1)(\gamma^2+1)^{7/2}-2(\gamma^2+p^2)^{7/2}-2(p^2\gamma^2+1)^{7/2}\right] \div\left[2p^7+7p^5+(8\sqrt{2}-9)(p^4+p^3)+7p^2+2-2(p^2+1)^{7/2}\right]$ |
| | $\tilde{h}_e(\gamma)$ | $\left[4p^3(\gamma^2+1)^{5/2}-2(p^2\gamma^2+1)^{5/2}-2(\gamma^2+p^2)^{5/2} +2(p^5-2p^3+1)(\gamma^5+1)-5p^2(p-1)(\gamma^3+\gamma^2)\right] \div\left[2p^5+(8\sqrt{2}-9)p^3+5p^2+2-2(p^2+1)^{5/2}\right]$ |

$$q_e^* = 1+\sum_{i=1}^{\kappa}(q_{w_i}^*-q_{w_{i-1}}^*)(1-\frac{\xi_{i-1}}{\xi}) \quad (41)$$

$$q_{w_i}^* = \frac{q_{w_i}}{q_{w_0}} \quad (42)$$

$$\xi_i = \frac{x_i}{x_0} \quad (43)$$

$\mathcal{H}_{m_\kappa}$ and $\mathcal{H}_{e_\kappa}$ are given by recurrence relations as

$$\mathcal{H}_{m_\kappa} = \mathcal{H}_{m_{\kappa-1}}+(q_{w_\kappa}^*-q_{w_{\kappa-1}}^*) \times\left[(q_{w_\kappa}^*-q_{w_{\kappa-1}}^*)\gamma_\kappa^7+\tilde{h}_m(\gamma_\kappa)+\sum_{i=1}^{\kappa-1}(q_{w_i}^*-q_{w_{i-1}}^*)\gamma_i^7\tilde{h}_m(\gamma_\kappa/\gamma_i)\right] \quad (44)$$

$$\mathcal{H}_{e_\kappa} = \mathcal{H}_{e_{\kappa-1}}+(q_{w_\kappa}^*-q_{w_{\kappa-1}}^*) \times\left[(q_{w_\kappa}^*-q_{w_{\kappa-1}}^*)\gamma_\kappa^5+\tilde{h}_e(\gamma_\kappa)+\sum_{i=1}^{\kappa-1}(q_{w_i}^*-q_{w_{i-1}}^*)\gamma_i^5\tilde{h}_e(\gamma_\kappa/\gamma_i)\right] \quad (45)$$

with $\mathcal{H}_{m_0}=\mathcal{H}_{e_0}=1$, and algebraic expressions for $\tilde{h}_m(\cdot)$ and $\tilde{h}_e(\cdot)$ can be found in Table 2.

This first order ordinary differential equation, Eq. (35), is independent of $\tilde{\Phi}_\kappa$ and is, therefore, decoupled from Eq. (36). With the above initial condition, it represents a complete problem that can be solved numerically for all $\gamma_i^3$ in succession for $i$ from 1 to $\kappa$ with given uniform $q_{w_i}^*$ and Pr. Upon finding $\gamma_\kappa^3$, $\tilde{\Phi}_\kappa$ can be determined from Eq. (36). The modifying functions $\tilde{\phi}_i$ and $\tilde{\psi}_i$ can be subsequently obtained by considering Eqs. (39) and (38), respectively. Readers who are interested in the detailed derivations and complete expressions of the differential terms appearing in the above differential equation may find the full context in Lee, 1988, where additional discussions on the modifying functions are also included.

## RESULTS AND DISCUSSION

**Temperature and Velocity Distributions.** By substituting the $t-x$ transformations, defined by Eqs. (30) through (34), into Eqs. (9) and (10), a set of dimensionless approximate solutions to the original steady state problem may be obtained as

$$\theta = \frac{T}{q_{w_0}x_0/k\mathrm{Gr}_{x_0}^{*\;1/5}} = \frac{\xi^{1/5}\sqrt{\tilde{\Phi}_\kappa}}{\tilde{C}_\eta}\left[f_T(\eta_0)+\sum_{i=1}^{\kappa}(q_{w_i}^*-q_{w_{i-1}}^*)\gamma_i f_T(\eta_i)\right] \quad (46)$$

$$U = \frac{u}{\frac{\nu}{x_0}\mathrm{Gr}_{x_0}^{*\;2/5}} = \xi^{3/5}\tilde{C}_u\tilde{\Phi}_\kappa^{3/2}\left[f_u(\eta_0)+\sum_{i=1}^{\kappa}(q_{w_i}^*-q_{w_{i-1}}^*)\gamma_i^3 f_u(\eta_i)\right] \quad (47)$$

for $x_{\kappa-1}<x\leq x_\kappa$, where $f_T(\cdot)$ and $f_u(\cdot)$ are as defined by Eqs. (11) and (12), respectively, and

$$\eta_0=\gamma_i\eta_i=\frac{\tilde{C}_\eta}{\xi^{1/5}\sqrt{\tilde{\Phi}_\kappa}}\mathrm{Gr}_{x_0}^{*\;1/5}\frac{y}{x_0} \quad (48)$$

For an additional $\kappa$-th step, $\gamma_\kappa$ and $\tilde{\Phi}_\kappa$ found in the previous section are required to complete the above solutions.

The above expressions for the temperature and velocity distributions are given in dimensionless forms with the non-dimensionalizing parameters which are invariant of $x$. They are evaluated for the case in air with a number of thermal sources mounted flush with the surface of an adiabatic plate, and the results are plotted in Fig. 2. All eight sources shown in this figure have uniform surface heat flux of equal strength and size, with non-source spaces equal to the source size. The value of $\tilde{C}_\eta$ from Eq. (20) was used in evaluating local temperatures throughout the present study, as it was recommended by Lee and Yovanovich (1988) for Pr > 0.1.

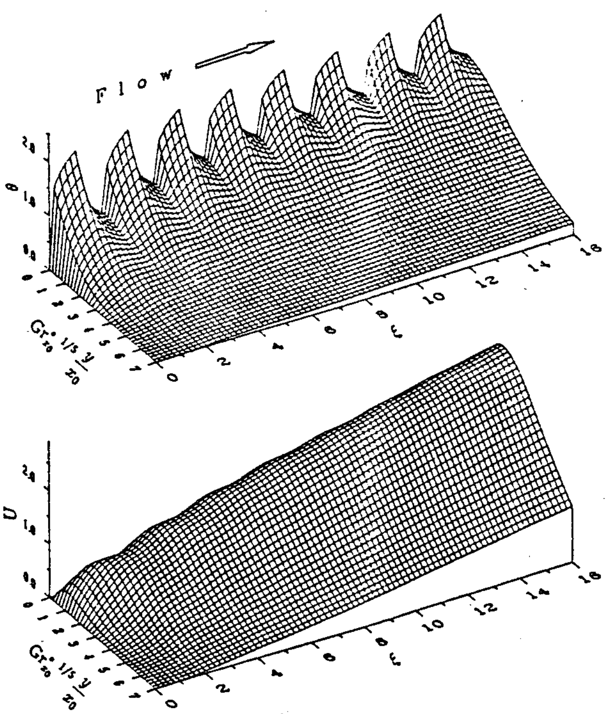


Figure 2: Dimensionless temperature and velocity distributions due to alternating positive uniform and zero surface heat fluxes of equal size : Pr = 0.7.

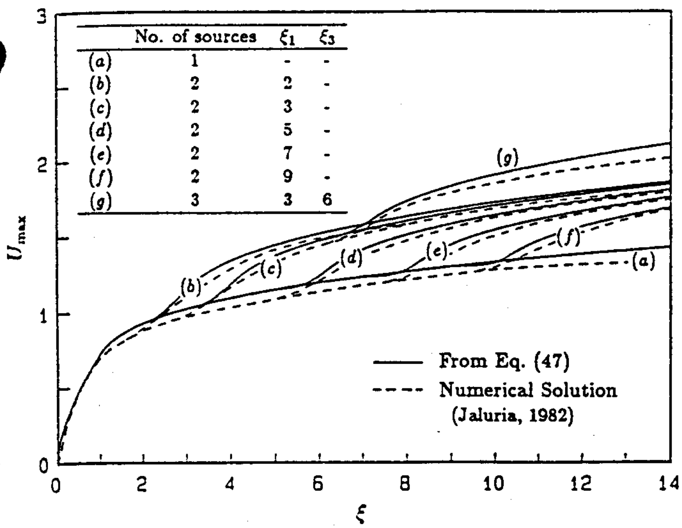


Figure 3: Comparison of maximum velocity variation with identical thermal sources of uniform surface heat flux on an adiabatic plate : $\mathrm{Pr} = 0.7$.

The dimensionless local maximum flow velocity, $U_{\max}$, is plotted and compared with the numerical data of Jaluria (1982) in Fig. 3 for cases with identical thermal sources and different source spacings. The sources are again mounted flush on an adiabatic plate in air, and they all have uniform surface heat flux. The figure shows excellent agreement of the velocity variations with the numerical results.

**Wall Temperature Variation.** The local wall temperature for $x_{\kappa-1} < x \leq x_\kappa$ can be obtained by substituting $\eta_0 = \eta_i = 0$ into Eq. (46). The resulting expression is

$$\theta_w = \frac{T_w}{q_{w_0} x_0 / k \mathrm{Gr}^{*\;1/5}_{x_0}} = \frac{\xi^{1/5}}{\tilde{C}_\eta}\sqrt{\frac{\tilde{\Phi}_\kappa}{\pi}}\left[1 + \sum_{i=1}^{\kappa}(q^*_{w_i} - q^*_{w_{i-1}})\gamma_i\right] \tag{49}$$

where $\mathrm{ierfc}\,0 = 1/\sqrt{\pi}$ was used.

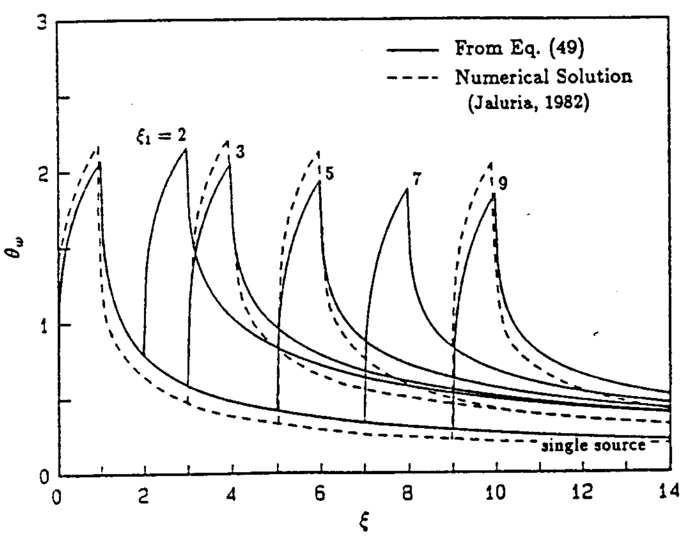


Figure 4: Comparison of dimensionless wall temperature variation due to two identical thermal sources of uniform surface heat flux on an adiabatic plate with various source spacings : $\mathrm{Pr} = 0.7$.

Comparisons of the wall temperature variations evaluated by using the present model are made with the results of Jaluria (1982) who used finite difference methods. The plots shown in Fig. 4 are obtained for two identical strip thermal sources of uniform surface heat flux, mounted flush on an adiabatic plate in air with a source spacing between the leading edges of the strips equal to $\xi_1$. Satisfactory agreement of the present results, particularly in the trend of the local peak temperatures with increasing $\xi_1$, with the numerical solutions is obtained. Figure 5 depicts a similar situation with three sources for $\mathrm{Pr} = 0.1$, 0.7 and 6, showing the effect of the Prandtl number on the surface temperature variations.

An examination of Fig. 4 reveals that the local peak temperature over the downstream source decays and becomes lower than that over the upstream source as the source spacing increases. The wall temperature due to the single source also decreases asymptotically to the ambient temperature as $\xi$ increases, and the fluid temperature excess within the boundary layer always remains positive.

The positive fluid temperature excess represents an upward buoyant force on the fluid. A portion of this force will be used to overcome the friction, and the remainder will be used to accelerate the flow, resulting in a perpetual increase in the overall downstream flow velocity. Despite the fact that there is a higher fluid temperature at the beginning of the second source than at the leading edge of the first one, this fluid flow results in a lower temperature over the second source when the source spacing is large enough. As far as the downstream source is concerned, the flow becomes an induced free stream flow and thus, enhances the heat transfer rate.

Jaluria's numerical prediction overestimates the peak temperature over the first source by approximately 5% when it is compared to the result of Eq. (49), whose overall scale of the magnitude is based on the correlation equation of Fujii and Fujii (1976) through $\tilde{C}_\eta$ from Eq. (20).

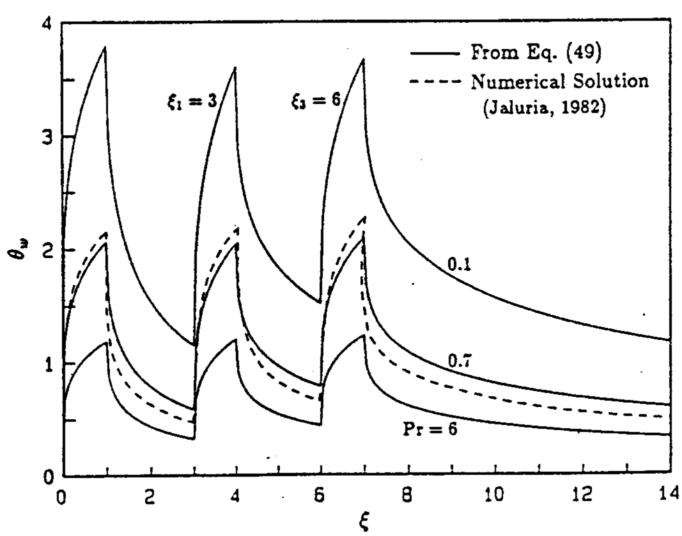


Figure 5: Comparison of dimensionless wall temperature variation due to three identical thermal sources of uniform surface heat flux on an adiabatic plate and the effect of Prandtl number.

The rest of this section exhibits some results obtained for cases that are intended to examine the validity and accuracy of the present model. A couple of figures for selected cases are presented herein for discussion purposes.

An expression for the dimensionless wall temperature variation based on an arbitrary reference heat flux $q_r$ and a reference length scale $x_r$ can be obtained from Eq. (49), as

$$\frac{T_w}{q_r x_r/k\mathrm{G}^{*1/5}} = \left(\frac{q_{w_0}}{q_r}\right)^{4/5}\left(\frac{x}{x_r}\right)^{1/5}\frac{1}{\tilde{C}_\eta}\sqrt{\frac{\tilde{\Phi}_\kappa}{\pi}}\left[1+\sum_{i=1}^{\kappa}(q^*_{w_i}-q^*_{w_{i-1}})\gamma_i\right] \tag{50}$$

where $\mathrm{G}^*$ is the modified Grashof number based on $q_r$ and $x_r$. This equation supersedes the previous equation for $\theta_w$, as they become identical when $q_{w_0}$ and $x_0$ are chosen to be the references. Although the above equation carries no additional information from Eq. (49), the usefulness of this expression is apparent in Fig. 6, as it allows to compare different wall temperature variations for various combinations of $q_{w_0}$ and $x_0$ on the same plot.

In Fig. 6, the line denoted by (*a*) is the result of alternating thermal sources of equal size dissipating at two different levels of uniform heat fluxes. The line denoted by (*b*) is for the case with a continuous and uniform surface heat flux whose magnitude is equal to the average value of the alternating heat fluxes. The lines denoted by (*c*) through (*f*) are the results of other combinations of heat flux distributions as shown at the top of the figure. All cases dissipate the same total amount of energy into the fluid over the length of the plate. They are evaluated for Pr = 0.7.

As can be seen from Fig. 6, the effect of the step changes in the heat flux input made at the upstream decays rather quickly, and it may be considered to penetrate no more than approximately five to six step distances into the downstream. The flow characteristics far enough downstream become indifferent to the specifics of the upstream heat input variation, as long as the total energy dissipated into the flow over the upstream section is kept the same. More examinations with different combinations of surface heat flux exhibited the same behavior.

Further testing, which involves another model, named a temperature model and developed by Lee (1988) for cases with a step change in surface temperature, is carried out as follows.

A step change in uniform wall temperature is denoted by (*a*) in Fig. 7 for $T^*_{w_1} = T_{w_1}/T_{w_0} = 0.75$ and Pr = 0.7. Using this as an input to the temperature model, the corresponding surface heat flux variation denoted by (*b*) is obtained. This surface heat flux variation is then discretized to yield multi-step changes denoted by (*c*). Although the step sizes can be arbitrary, they are determined such that every step before $x/x_r = 1$ dissipates roughly the same amount of energy into the fluid. The step sizes after $x/x_r = 1$ are determined simply by overlapping those from $x/x_r \leq 1$. The magnitude of heat flux in each step is equal to the average heat flux of the original variation over the step. This multi-step change in surface heat flux is used in the present model, Eq. (49), to produce the wall temperature variation denoted by (*d*). The reference heat flux, $q_r$, was sized to yield $T_{w_0}/(q_r x_r/k\mathrm{Gr}^{1/4}_{x_r}) = 1$ at the beginning of the process.

As can be seen from Fig. 7, excellent agreement between the input and the resulting temperature variations, lines (*a*) and (*d*), is obtained. Peaks observed in the resulting temperature variation at the leading edge, $x/x_r = 0$, and at the point of a step, $x/x_r = 1$, are due to the finite representation, line (*c*), of the unbounded original heat flux variation, line (*b*), at the locations. Performing the same process with other values of $T^*_{w_1}$ as an input wall temperature variation resulted in the same level of agreement.

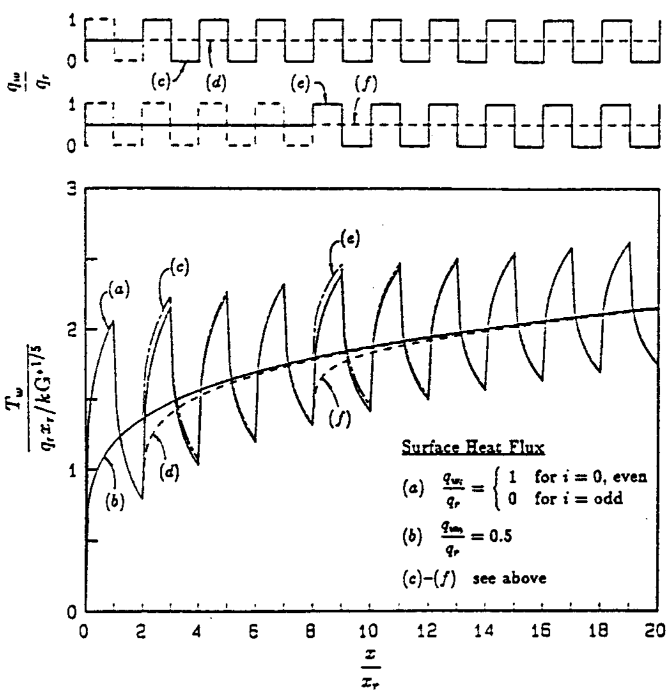


Figure 6: Dimensionless wall temperature variation : Pr = 0.7.

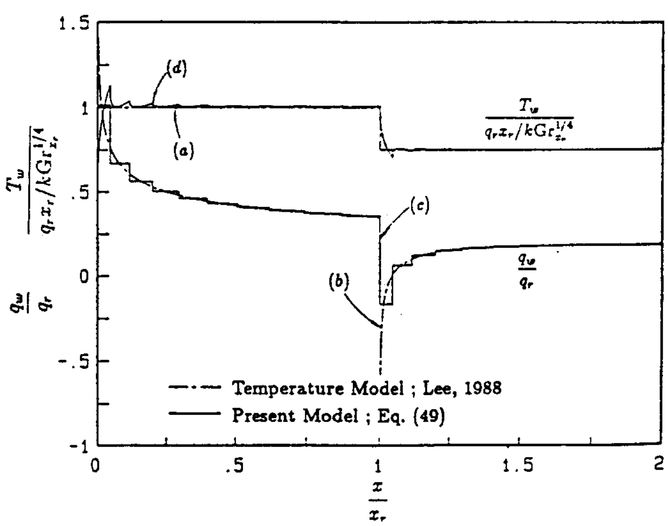


Figure 7: Dimensionless wall temperature and surface heat flux variations : Pr = 0.7.

Results of the present model are less accurate in the vicinity of the leading edge and locations of a discontinuity in surface thermal variations, due to the boundary layer approximations. Jaluria (1985), using finite difference methods, investigated this subject and presented the level of inaccuracies introduced by the approximations in both temperature and maximum velocity variations for different modified Grashof numbers.

A pertinent application of the model includes the thermal design of printed circuit boards (PCBs). In view of PCB applications, conjugate analyses would be required which include conduction heat transfer in the board substrate. When air is used as the coolant fluid as in most cases, the thermal conductivity of a typical substrate is usually orders of magnitude greater than that of the air, and the natural convection heat transfer coefficients are usually small. As a result, the in-plane conduction heat transfer would be characterized predominantly by the thermal properties of the substrate, and the effect of the fluid conduction heat transfer in the directions parallel to the plate surface may be neglected. This further supports the boundary layer approximation in the energy equation which already ignores the streamwise conduction heat transfer in the fluid.

Since the model is capable of dealing with arbitrary surface heat flux variations, it can be used in conjunction with a heat conduction analysis in the solid plate for a two dimensional flat plate modeling of conjugate heat transfer problems (Lee and Yovanovich, 1989). Further, as the fluid conduction in the direction across the plate width may be neglected as aforementioned, the present two dimensional model may also be used in solving full three dimensional conjugate problems involving an isolated vertical flat plate.

## ACKNOWLEDGEMENTS

The authors wish to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada under a Postgraduate Scholarship for Dr. Lee, and the operating grant to Dr. Yovanovich.

## REFERENCES

**Bar-Cohen, A., 1985,** "Bounding Relations for Natural Convection Heat Transfer from Vertical Printed Circuit Boards," *Proceedings of IEEE,* Vol. 73, No. 9, pp. 1388–1395.

**Fujii, T., and Fujii, M., 1976,** "The Dependence of Local Nusselt Number on Prandtl Number In the Case of Free Convection Along a Vertical Surface With Uniform Heat Flux," *Int. J. Heat Mass Transfer,* Vol. 19, pp. 121–122.

**Jaluria, Y., 1982,** "Buoyancy-Induced Flow Due to Isolated Thermal Sources on a Vertical Surface," *J. Heat Transfer,* Vol. 104, pp. 223–227.

**Jaluria, Y., 1985,** "Interaction of Natural Convection Wakes Arising From Thermal Sources on a Vertical Surface," *J. Heat Transfer,* Vol. 107, pp. 883–892.

**Jaluria, Y., and Gebhart, B., 1977,** "Buoyancy-Induced Flow Arising From a Line Thermal Source on an Adiabatic Vertical Surface," *Int. J. Heat Mass Transfer,* Vol. 20, pp. 153–157.

**Lee, S., 1988,** "Laminar Natural Convection From a Vertical Plate With Variations in Thermal Boundary Conditions," Ph.D. Thesis, Department of Mechanical Engineering, University of Waterloo, Ontario.

**Lee, S., and Yovanovich, M.M., 1988,** "Laminar Natural Convection From a Vertical Plate With Variations in Surface Heat Flux," ASME HTD-Vol. 96, ed. H. R. Jacobs, Vol. 2, pp. 197–205, The 25th National Heat Transfer Conf., July 24–27, Houston, Texas.

**Lee, S., and Yovanovich, M.M., 1989,** "Conjugate Heat Transfer From a Vertical Plate With Discrete Heat Sources Under Natural Convection," - submitted for presentation at the ASME Winter Annual Meeting, December 10–15, San Francisco, California.

**Ostrach, S., 1953,** "An Analysis of Laminar Free-Convection Flow and Heat Transfer About a Flat Plate Parallel to the Direction of the Generating Body Force," NACA Report No. 1111.

**Raithby, G.D., and Hollands, K.G.T., 1975,** "A General Method of Obtaining Approximate Solutions to Laminar and Turbulent Free Convection Problems," *Advances in Heat Transfer,* Vol. 11, pp. 265–315, Academic Press, NY.

**Schetz, J.A., 1963,** "On the Approximate Solution of Viscous-Flow Problems," *J. Appl. Mech.,* Vol. 30, *Trans. ASME,* Vol. 85, pp. 263–268.

**Sparrow, E.M., and Gregg, J.L., 1956,** "Laminar Free Convection From a Vertical Plate With Uniform Surface Heat Flux," *Trans. ASME,* Vol. 78, pp. 435–440.

**Sparrow, E.M., and Gregg, J.L., 1958,** "Similar Solutions for Free Convection From a Nonisothermal Vertical Plate," *Trans. ASME,* Vol. 80, pp. 379–386.

**Tribus, M., 1958,** "Discussion on Similar Solutions for Free Convection From a Nonisothermal Vertical Plate," *Trans. ASME,* Vol. 80, pp. 1180-1181.

**Yang, K.T., 1960,** "Possible Similarity Solutions for Laminar Free Convection on Vertical Plates and Cylinders," *Trans. ASME,* Vol. 82, pp. 230–236.

**Zinnes, A.E., 1970,** "The Coupling of Conduction With Laminar Natural Convection From a Vertical Flat Plate With Arbitrary Surface Heating," *J. Heat Transfer,* Vol. 92, pp. 528–535.