

THERMAL SPREADING RESISTANCES IN RECTANGULAR FLUX CHANNELS PART I - GEOMETRIC EQUIVALENCES

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ABSTRACT

This paper presents a simple geometric approach to predicting thermal spreading resistance in isotropic and compound rectangular flux channels using the solution for an isotropic or compound circular flux tube. It is shown that the results are valid for a wide range of channel aspect ratios and source to base coverage ratio. Since the circular disk solution requires a single series summation it is preferable to the rectangular flux channel solution which requires the evaluation of two single series and one double series summation.

Keywords: Conduction, Spreading Resistance, Heat Spreaders, Contact Heat Transfer, Electronic Packaging

NOMENCLATURE

a, b	=	radial dimensions, m
a, b, c, d	=	linear dimensions, m
A_b	=	baseplate area, m^2
A_s	=	heat source area, m^2
A_n, B_n	=	Fourier coefficients
Bi	=	Biot number, $h\mathcal{L}/k$
h	=	contact conductance or film coefficient, $W/m^2 \cdot K$

$J_0(\cdot), J_1(\cdot)$	=	Bessel functions of first kind
k	=	thermal conductivity, $W/m \cdot K$
$\mathcal{L} = \sqrt{A_s}$	=	arbitrary length scale, m
m, n	=	indices for summations
Q	=	heat flow rate, $\equiv q_s A_s, W$
q_s	=	heat flux, W/m^2
R	=	thermal resistance, K/W
R_{1D}	=	one-dimensional resistance, K/W
R_s	=	spreading resistance, K/W
R_t	=	total resistance, K/W
R^*	=	dimensionless resistance, $\equiv kR\mathcal{L}$
t, t_1, t_2	=	total and layer thicknesses, m
T	=	temperature, K
\bar{T}_s	=	mean source temperature, K
T_f	=	sink temperature, K

Greek Symbols

α	=	equation parameter, $\equiv (1 - \kappa)/(1 + \kappa)$
β_{mn}	=	eigenvalues, $\equiv \sqrt{\delta_m^2 + \lambda_n^2}$
δ_m	=	eigenvalues, $(m\pi/c)$
ϵ	=	relative source size, $\equiv a/b$
ϵ_x	=	relative source size, $\equiv a/c$
ϵ_y	=	relative source size, $\equiv b/d$
ϵ_b	=	baseplate aspect ratio, $\equiv c/d$
θ	=	temperature excess, $\equiv T - T_f, K$
$\bar{\theta}$	=	mean temperature excess, $\equiv \bar{T} - T_f, K$
κ	=	relative conductivity, k_2/k_1
λ_n	=	eigenvalues, $(n\pi/d)$
ϕ, φ	=	spreading resistance functions
ψ	=	spreading parameter, $4kaR_s$
ϱ	=	equation parameter, $\equiv (\zeta + h/k_2)/(\zeta - h/k_2)$

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τ = relative thickness, $\equiv t/\mathcal{L}$
 ζ = dummy variable, m^{-1}

Subscripts

b = base
 e = equivalent
 m, n = m^{th} and n^{th} terms
 s = source
 t = total
 x = x-dir
 y = y-dir

INTRODUCTION

Thermal spreading resistance in rectangular flux channels is of interest to electronic packaging engineers working with discrete heat sources in heat sink, circuit board and a host of other applications where heat enters through a portion of the contacting surface.^{1,2} In the first part of this paper, geometric equivalences between the circular disk and rectangular flux channel are established. A review of the literature shows that a number of useful solutions for rectangular flux channels have been obtained for a variety of configurations.³⁻⁷ However, these solutions which are based on Fourier series expansions,^{8,9} require the evaluation of single and double summations. Although these solutions are not computationally intractable in the present day, a solution requiring the evaluation of a single series is much more efficient. Such is the case for thermal spreading resistances in circular flux tubes.^{10,11} This paper will demonstrate that a geometric equivalence may be obtained for predicting thermal spreading resistances in rectangular flux channels using the solution for an equivalent circular flux tube. It will be shown that the solution is a weak function of shape and aspect ratio of the heat source and substrate. Theoretical results will be presented for a range of parameters and simple expressions will be developed to assist in the computations.

PROBLEM STATEMENT

Thermal spreading resistance arises in multi-dimensional applications where heat enters a domain through a finite area. In typical applications, the system is idealized as having a central heat source placed on one of the heat spreader surfaces, while the lower surface is cooled with a constant conductance which may represent a heat sink, contact conductance, or convective heat transfer. All edges are assumed to be adiabatic as is the region outside of the heat source on the source plane.

In this idealized system, the total thermal resistance of the system may be defined as

$$R_t = \frac{\bar{T}_s - T_f}{Q} = \frac{\bar{\theta}_s}{Q} \quad (1)$$

where θ_s is the mean source temperature excess and Q is the total heat input of the device. The mean source temperature is given by

$$\bar{\theta}_s = \frac{1}{A_s} \iint_{A_s} \theta(x, y, 0) dA_s \quad (2)$$

In applications involving adiabatic edges, the total thermal resistance is composed of two terms: a uniform flow or one-dimensional resistance and a spreading or multi-dimensional resistance, which vanishes as the source area approaches the substrate area. These two components are combined as follows:

$$R_t = R_{1D} + R_s \quad (3)$$

Thermal spreading resistance analysis of a rectangular spreader requires the solution of Laplace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (4)$$

in three dimensions, and for circular disk spreaders

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (5)$$

in two dimensions.

In most applications the following boundary conditions are applied:

$$\frac{\partial T}{\partial n} = 0 \quad n = x, y, r \quad (6)$$

along the edges, $x = c, y = d$ or $r = b$, and at the centroid of the substrate, $x = 0, y = 0$ or $r = 0$. Over the top surface $z = 0$,

$$\left. \begin{aligned} \frac{\partial T}{\partial z} &= 0, & A_s < A < A_b \\ \frac{\partial T}{\partial z} &= -\frac{q_s}{k}, & 0 < A < A_s \end{aligned} \right\} \quad (7)$$

where A_s is the area of the heat source and A_b is the area of the base or substrate. Finally, along the lower surface $z = t$,

$$\frac{\partial T}{\partial z} + \frac{h}{k}(T - T_f) = 0 \quad (8)$$

where h is a uniform convection heat transfer coefficient or contact conductance.

In compound systems, Laplace's equation must be written for each layer in the system, and continuity of temperature and heat flux at the material interface is required, yielding two additional boundary conditions:

$$\left. \begin{aligned} T_1 &= T_2 \\ k_1 \frac{\partial T_1}{\partial z} &= k_2 \frac{\partial T_2}{\partial z} \end{aligned} \right\} \quad (9)$$

For a compound system Eq. (8) is written to represent T_2 , i.e.,

$$\frac{\partial T_2}{\partial z} + \frac{h}{k}(T_2 - T_f) = 0 \quad (10)$$

Due to the nature of the solution, the total thermal resistance may be analyzed as two problems. One is steady one-dimensional conduction which yields the uniform flow component of the thermal resistance, while the other is a multi-dimensional conduction analysis using Fourier series or finite integral transform methods to solve an eigenvalue problem. This paper is mainly concerned with modeling the solution to the thermal spreading resistance component in systems with one or two layers. The necessary solutions for the systems to be examined are found in the papers by Yovanovich et al.^{3,10} They are given below for the sake of completeness since they will be non-dimensionalized in a more appropriate manner.

Cylindrical Systems

Thermal spreading resistance solutions in isotropic and compound disks, (refer to Figs. 1 and 2), flux channels and half spaces are presented in Yovanovich et al.^{10,11} A general solution for the compound disk was first obtained by Yovanovich et al.¹¹ The general solution¹¹ is

$$4k_1 a R_s = \frac{8}{\pi \epsilon} \sum_{n=1}^{\infty} A_n(n, \epsilon) B_n(n, \tau, \tau_1) \frac{J_1(\delta_n \epsilon)}{\delta_n \epsilon} \quad (11)$$

where

$$A_n = -\frac{2\epsilon J_1(\delta_n \epsilon)}{\delta_n^2 J_0^2(\delta_n)} \quad (12)$$

and

$$B_n = \frac{\phi_n \tanh(\delta_n) - \varphi_n}{1 - \phi_n} \quad (13)$$

The functions ϕ_n and φ_n are defined as follows:

$$\phi_n = \frac{\kappa - 1}{\kappa} \cosh(\delta_n \tau_1) [\cosh(\delta_n \tau_1) - \varphi_n \sinh(\delta_n \tau_1)] \quad (14)$$

and

$$\varphi_n = \frac{\delta_n + Bi \tanh(\delta_n \tau)}{\delta_n \tanh(\delta_n \tau) + Bi} \quad (15)$$

The eigenvalues δ_n are solutions to $J_1(\delta_n) = 0$ and $Bi = hb/k_2$, $\tau = t/b$, and $\tau_1 = t_1/b$.

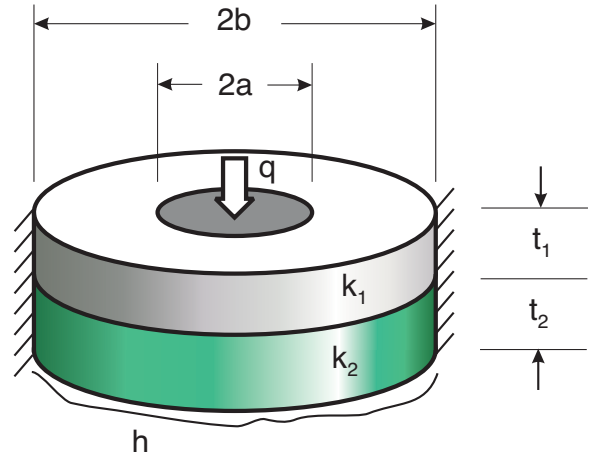


Fig. 1 - Compound Disk with Circular Heat Source

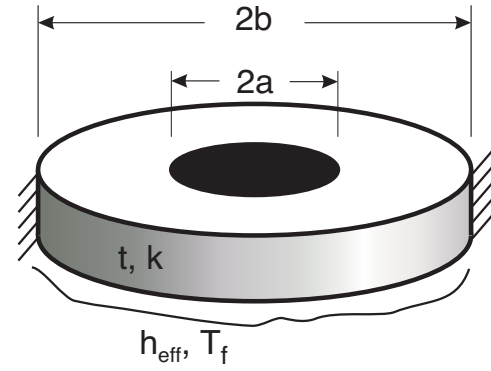


Fig. 2 - Isotropic Disk with Circular Heat Source

Rectangular Systems

Thermal spreading resistance in rectangular systems was recently obtained by the authors.³ In Yovanovich et al.³, the authors obtained a solution for a compound rectangular flux tube having a central heat source, (refer to Fig. 3). This general solution also simplifies for many cases of semi-infinite flux channels and half space solutions.³ More recently, the authors⁴ developed a solution for a single eccentric heat source on compound and isotropic flux channels. The results of Muzychka et al.⁴ were also extended to systems having multiple arbitrarily located heat sources.

The spreading resistance of Yovanovich et al.³ is obtained from the following general expression according to the notation in Figs. 3 and 4:

$$R_s = \frac{c^2}{2k_1 a^2 d} \sum_{m=1}^{\infty} \frac{\sin^2(m\pi a/c)}{(m\pi)^3} \cdot \varphi(\delta_m)$$

$$\begin{aligned}
& + \frac{d^2}{2k_1 b^2 c} \sum_{n=1}^{\infty} \frac{\sin^2(n\pi b/d)}{(n\pi)^3} \cdot \varphi(\lambda_n) \\
& + \frac{cd}{k_1 a^2 b^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi a/c) \sin^2(n\pi b/d)}{(m\pi)^2 (n\pi)^2 \beta_{mn}} \cdot \varphi(\beta_{mn})
\end{aligned} \tag{16}$$

where

$$\varphi(\zeta) = \frac{(\alpha e^{4\zeta t_1} + e^{2\zeta t_1}) + \varrho (e^{2\zeta(2t_1+t_2)} + \alpha e^{2\zeta(t_1+t_2)})}{(\alpha e^{4\zeta t_1} - e^{2\zeta t_1}) + \varrho (e^{2\zeta(2t_1+t_2)} - \alpha e^{2\zeta(t_1+t_2)})} \tag{17}$$

and

$$\varrho = \frac{\zeta t + ht/k_2}{\zeta t - ht/k_2} \quad \text{and} \quad \alpha = \frac{1 - \kappa}{1 + \kappa}$$

with $\kappa = k_2/k_1$. The eigenvalues for these solutions are: $\delta_m = m\pi/c, \lambda_n = n\pi/d$ and $\beta_{mn} = \sqrt{\delta_m^2 + \lambda_n^2}$ and are denoted by ζ in Eq. (17). Equation (17) simplifies for an isotropic disk to give

$$\varphi_n = \frac{\zeta t + ht/k_1 \tanh(\zeta t)}{\zeta t \tanh(\zeta t) + ht/k_1} \tag{18}$$

where k_1 is now the thermal conductivity of the flux channel.

In the next section these two solutions are compared, and it is shown that considerable computational effort is saved by modeling the rectangular flux channel as an equivalent circular flux tube for a wide range of channel aspect ratios.

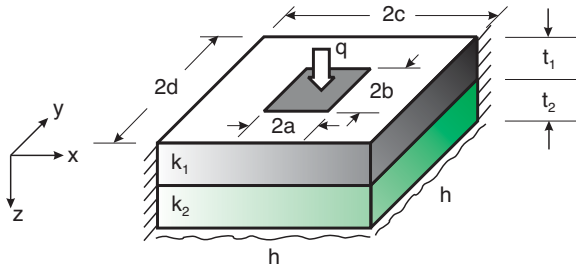


Fig. 3 - Compound Flux Channel with Rectangular Heat Source

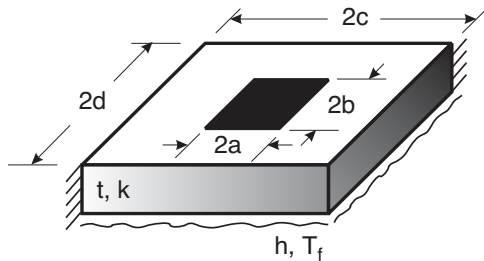


Fig. 4 - Isotropic Flux Channel with Rectangular Heat Source

ANALYSIS

Given the two solutions for the circular disk and rectangular flux channel, it is now possible to show that there exists a geometric equivalence between the two systems as shown in Fig. 5. Geometric equivalence was established by Muzychka et. al.¹² for computing the spreading resistance in annular sectors. In determining this equivalence, we choose to maintain the volume of material in the system. In doing so, the following conditions must be satisfied:

$$\left. \begin{aligned} A_{s,R} &= A_{s,C} \\ A_{b,R} &= A_{b,C} \\ t_R &= t_C \end{aligned} \right\} \tag{19}$$

or

$$\left. \begin{aligned} a_e &= \sqrt{A_s/\pi} \\ b_e &= \sqrt{A_b/\pi} \\ t &= t \end{aligned} \right\} \tag{20}$$

This is equivalent to non-dimensionalizing the circular disk solution using the characteristic length scale $\mathcal{L} = \sqrt{A_s} = \sqrt{\pi a}$. This length scale has been examined in the past in references.¹³⁻¹⁵

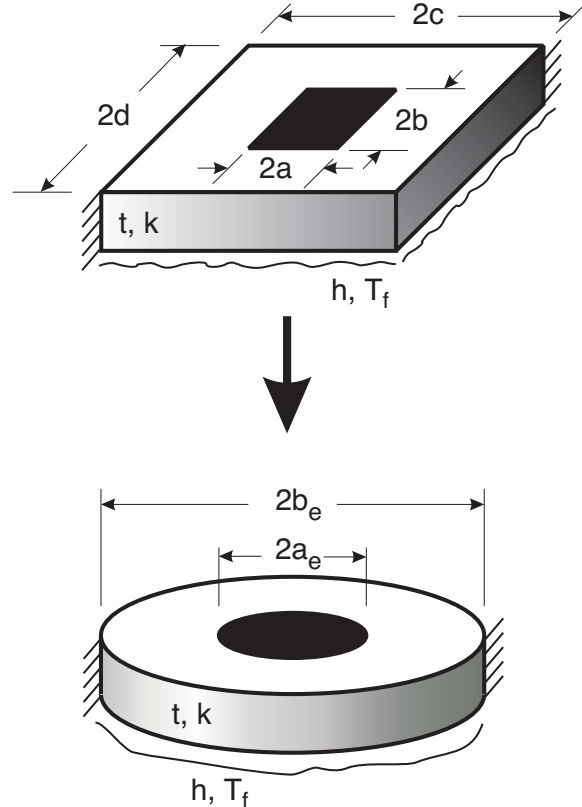


Fig. 5 - Equivalence Between a Rectangular Flux Channel and Circular Flux Tube

This leads to the following dimensionless spreading resistance defined as $R_s^* = k\sqrt{A_s}R_s$ for an isotropic circular flux tube:

$$R_s^* = \frac{4}{\sqrt{\pi}\epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)} \cdot \phi_n \quad (21)$$

where

$$\phi_n = \frac{\delta_n \epsilon \sqrt{\pi} + Bi \tanh(\delta_n \epsilon \sqrt{\pi} \tau)}{Bi + \delta_n \epsilon \sqrt{\pi} \tanh(\delta_n \epsilon \sqrt{\pi} \tau)} \quad (22)$$

The one dimensional resistance may be written in dimensionless form as:

$$R_{1D}^* = \left(\tau + \frac{1}{Bi} \right) \epsilon^2 \quad (23)$$

Thus, the spreading, one dimensional, and total resistances are a function of

$$R^* = f(\epsilon, \tau, Bi) \quad (24)$$

where $\tau = t/\sqrt{A_s}$, $Bi = h\sqrt{A_s}/k$, and $\epsilon = \sqrt{A_s/A_b} = a/b$.

The solution for spreading resistance in a rectangular flux channel may be non-dimensionalized in a similar manner using $\mathcal{L} = \sqrt{A_s} = 2\sqrt{ab}$. The resulting expression may be written as follows:

$$R_s^* = \sum_{m=1}^{\infty} \frac{\sin^2(m\pi\epsilon_x) \sqrt{\epsilon_x \epsilon_y \epsilon_b}}{(m\pi)^3 \epsilon_x^2} \cdot \phi_x + \sum_{n=1}^{\infty} \frac{\sin^2(n\pi\epsilon_y) \sqrt{\epsilon_x \epsilon_y \epsilon_b}}{(n\pi)^3 \epsilon_y^2 \epsilon_b} \cdot \phi_y + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi\epsilon_x) \sin^2(n\pi\epsilon_y) \sqrt{\epsilon_x \epsilon_y \epsilon_b}}{\epsilon_x^2 \epsilon_y^2 m^2 n^2 \pi^5 \sqrt{m^2/\epsilon_s^2 + n^2}} \cdot \phi_{xy} \quad (25)$$

The spreading functions ϕ which account for the effects of the film coefficient h and finite thickness t are:

$$\phi(\xi) = \frac{\xi + Bi \tanh(\xi\tau)}{Bi + \xi \tanh(\xi\tau)} \quad (26)$$

where

$$\left. \begin{aligned} \phi_x &= \phi(\xi \rightarrow 2\pi m \sqrt{\epsilon_x \epsilon_y \epsilon_b} / \epsilon_b) \\ \phi_y &= \phi(\xi \rightarrow 2\pi n \sqrt{\epsilon_x \epsilon_y \epsilon_b}) \\ \phi_{xy} &= \phi(\xi \rightarrow 2\pi \sqrt{m^2/\epsilon_b^2 + n^2} \sqrt{\epsilon_x \epsilon_y \epsilon_b}) \end{aligned} \right\} \quad (27)$$

The one dimensional resistance may be written in dimensionless form as:

$$R_{1D}^* = \left(\tau + \frac{1}{Bi} \right) \epsilon_x \epsilon_y \quad (28)$$

Thus, the dimensionless spreading, one dimensional, and total resistances are now a function of

$$R^* = f(\epsilon_x, \epsilon_y, \epsilon_b, \tau, Bi) \quad (29)$$

where $\epsilon_x = a/c$, $\epsilon_y = b/d$, $\epsilon_b = c/d$, $\tau = t/\sqrt{A_s}$, and $Bi = h\sqrt{A_s}/k$.

Similar results may be obtained for compound disks and flux channels. The comparisons in the next section will be made for isotropic systems, but the analysis is also valid for compound systems.

RESULTS AND DISCUSSION

We may now compare the solution for the disk and the flux channel using Eqs. (21) and (25). Two configurations are examined. These are the square flux channel with a central square heat source, and a rectangular flux channel with a rectangular heat source which may or may not conform to the aspect ratio of the flux channel as shown in Fig. 6. In the case of the flux tube solution 200 terms were used in the summation in Eq. (21). In the case of the flux channel, 200 terms were used in each of the single summations and 50 terms in the double summation of Eq. (25). This provided accuracy to at least four decimal places.

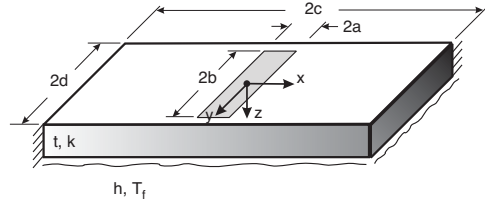


Fig. 6 - Non-conforming Source/Substrate System

First, for the special case of a square flux channel with a square heat source, the solution simplifies considerably, since $\epsilon_b = 1$, and $\epsilon_x = \epsilon_y = \epsilon$. The solutions for both the circular disk and square flux channel are given in Figs. 7-11 for a range of dimensionless thickness τ , Bi , and ϵ . Excellent agreement is obtained for these two cases.

Next we examine the effect of flux channel and heat source aspect ratios. Three flux channel aspect ratios are examined: $\epsilon_b = 1 : 1, 2 : 1, 4 : 1$. For each of these cases, the source aspect ratio is varied such that $\epsilon_x = 0.2, 0.4, 0.6, 0.8$ and $\epsilon_y = 0.2, 0.4, 0.6, 0.8$. This leads to sixteen combinations for each flux channel considered. Further, we have also considered three dimensionless thicknesses: $\tau = 0.01, 0.1, 1$ and two Biot numbers: $Bi = 10, 100$. In order to compare the circular flux tube and the rectangular flux channel, the results must be plotted using the common aspect ratio defined as:

$$\epsilon = \sqrt{A_s/A_b} = \sqrt{\epsilon_x \epsilon_y} \quad (30)$$

Thus for each case examined an equivalent circular aspect ratio is determined from ϵ_x and ϵ_y . This leads to nine unique predictions using the flux tube solution. Therefore, for some aspect ratios two solutions

are determined for the flux channel due to source orientation, while there is only one equivalent flux tube solution. The results of this comparison are presented in Figs. 12-17. It is clear that for most combinations with $\tau < 1$, there exists an equivalence between the two systems for which the error is less than ± 10 percent. The accuracy increases as Bi increases and as τ decreases for all values of ϵ considered.

SUMMARY AND CONCLUSIONS

This paper examined the exact solutions of the circular flux tube and rectangular flux channel for both isotropic and compound systems. It was shown that the solution for the circular flux tube may be used to model the rectangular flux channel when an appropriate geometric equivalence is established. Graphical results were presented for a wide range of parameters. It was shown that the equivalence is accurate for moderately sized contacts of any aspect ratio. These results will be utilized in Part II to facilitate an equivalence in similar systems with edge cooling.

ACKNOWLEDGMENTS

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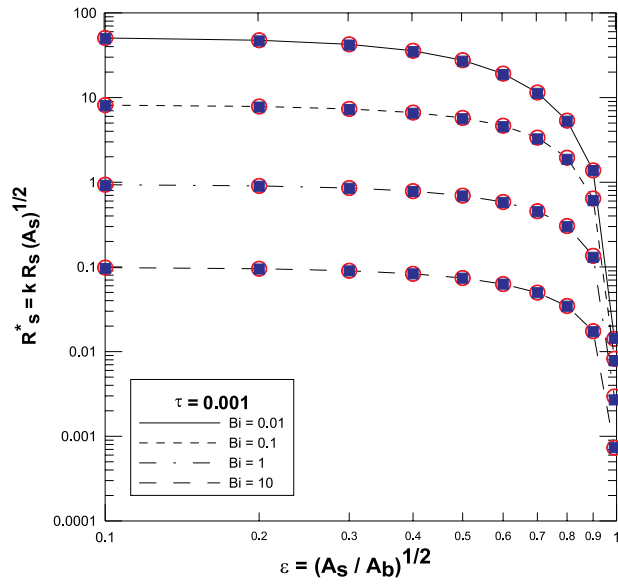


Fig. 7 - Equivalence for a Square Channel for $\tau = 0.001$.

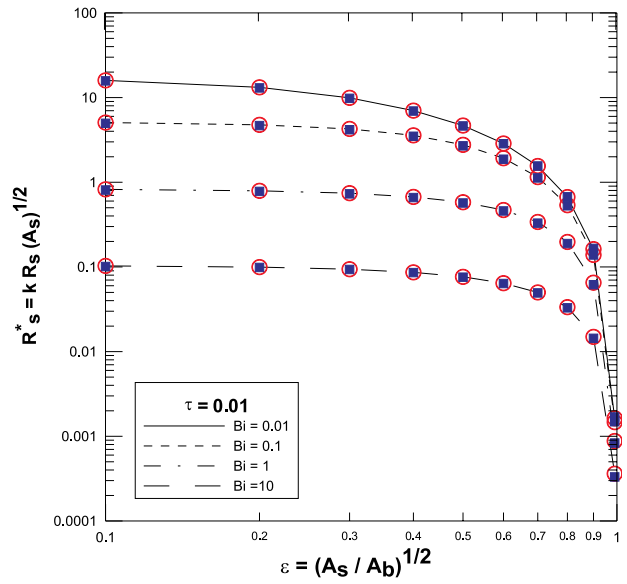


Fig. 8 - Equivalence for a Square Channel for $\tau = 0.01$.

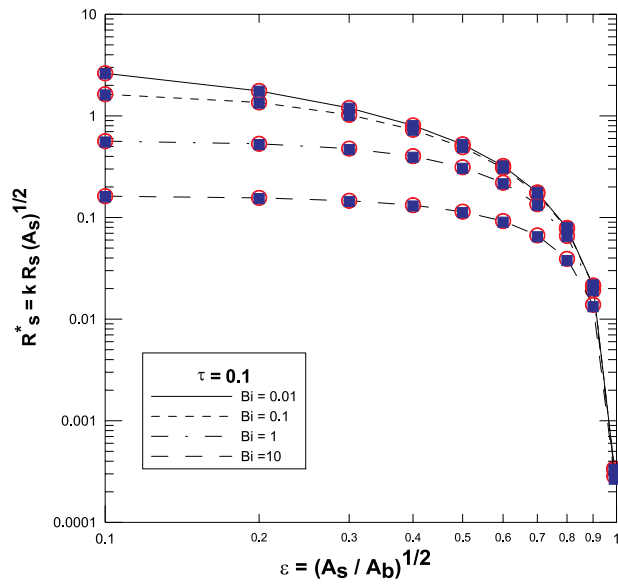


Fig. 9 - Equivalence for a Square Channel for $\tau = 0.1$.

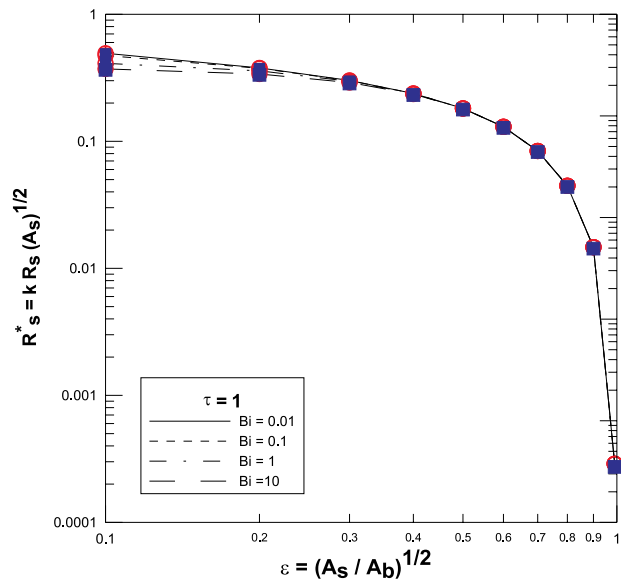


Fig. 10 - Equivalence for a Square Channel for $\tau = 1$.

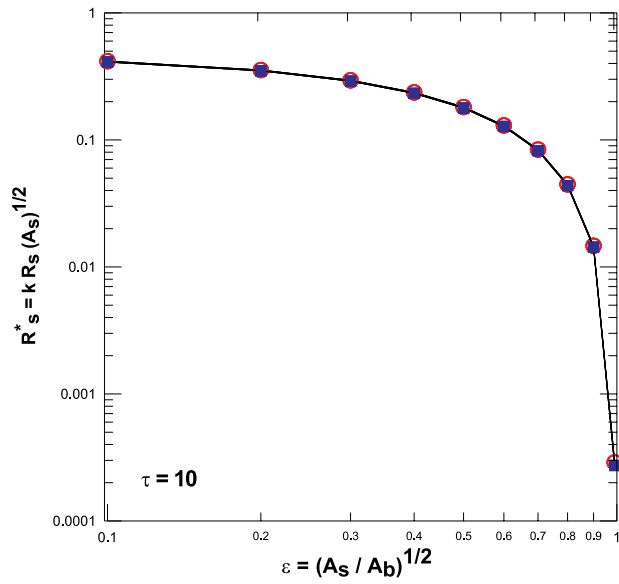


Fig. 11 - Equivalence for a Square Channel for $\tau = 10$.

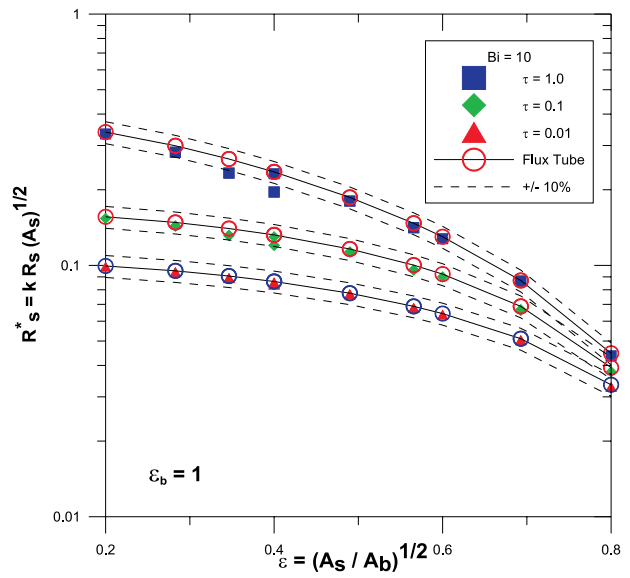


Fig. 12 - Equivalence Model for $Bi = 10$ and $\epsilon_b = 1$.

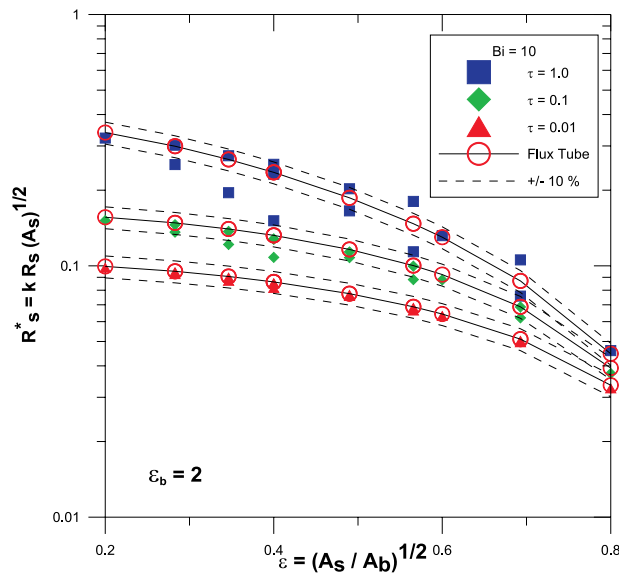


Fig. 13 - Equivalence Model for $Bi = 10$ and $\epsilon_b = 2$.

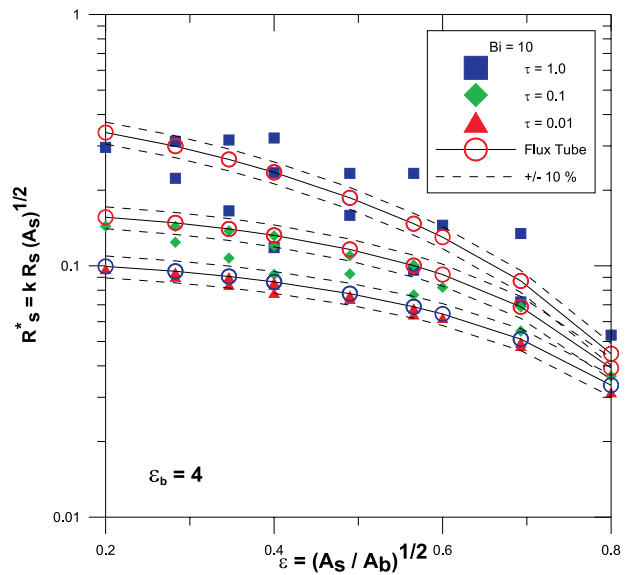


Fig. 14 - Equivalence Model for $Bi = 10$ and $\epsilon_b = 4$.

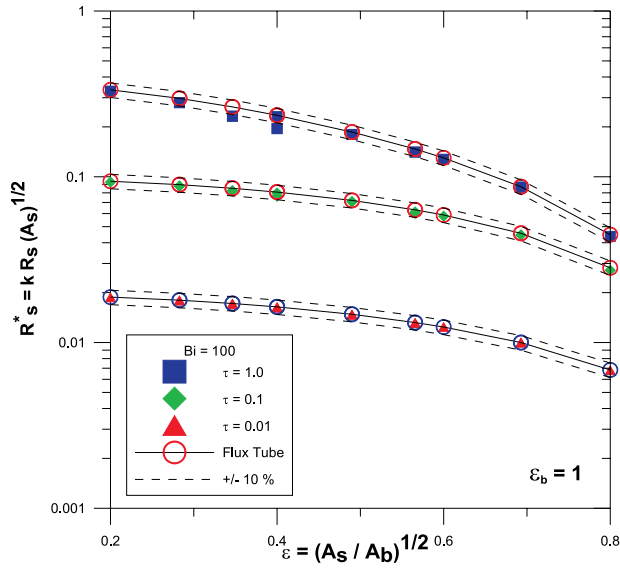


Fig. 15 - Equivalence Model for $Bi = 100$ and $\epsilon_b = 1$.

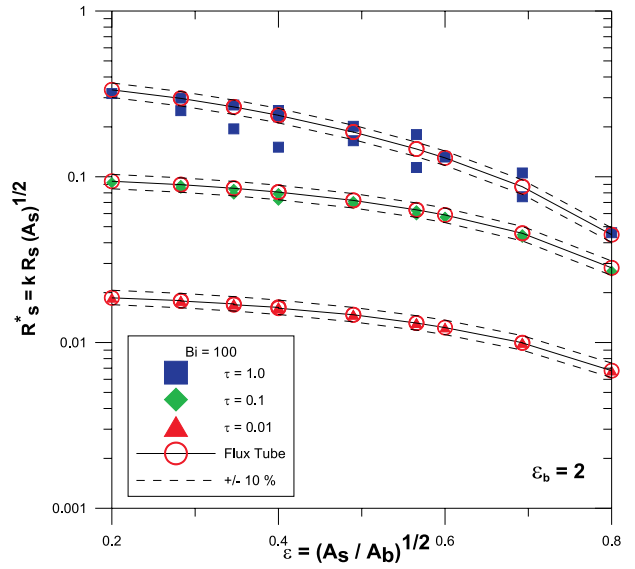


Fig. 16 - Equivalence Model for $Bi = 100$ and $\epsilon_b = 2$.

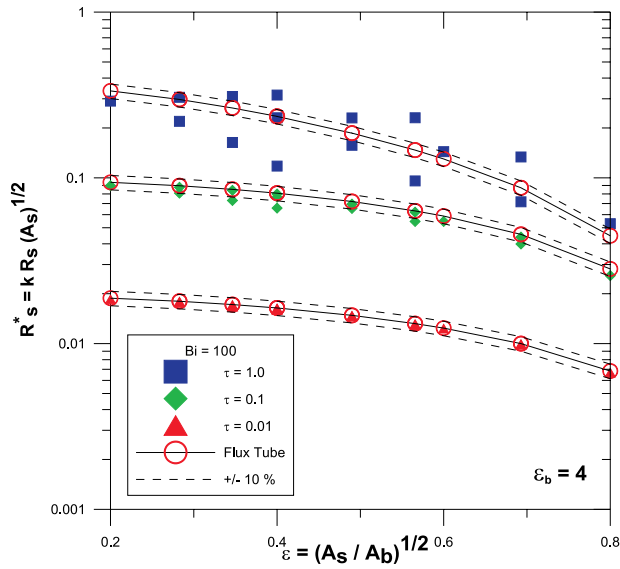


Fig. 17 - Equivalence Model for $Bi = 100$ and $\epsilon_b = 4$.