Comparison of Structured and Weighted Total Least-Squares Adjustment Methods for Linearly Structured Errors-in-Variables Models

Yongjun Zhou1; Xinjian Kou2; Jonathan Li3; and Xing Fang4

Abstract: The paper focuses on a specific errors-in-variables (EIV) model named the linearly structured EIV (LSEIV) model in which all the random elements of design matrix are in a linear combination of an input vector with random errors. Two existing structured total least-squares (STLS) algorithms named constrained TLS (CTLS) and structured TLS normalization (STLN) are introduced to solve the LSEIV model by treating the input and output vectors as the noisy structure vectors. For comparison purposes, the weighted TLS (WTLS) method is also performed based on the partial EIV model. Approximated accuracy assessment methods are also presented. The plane fitting and Bursa transformation examples are illustrated to demonstrate the accuracy and computational efficiency performance of the proposed algorithms. It shows that the proposed STLS and WTLS algorithms can achieve the same accuracy if the dispersion matrix of the WTLS method is constructed based on the partial EIV model. DOI: 10.1061/(ASCE)SU.1943-5428.0000190. © 2016 American Society of Civil Engineers.

Author keywords: Errors-in-variables (EIV) model; Structured total least-squares (STLS); Weighted total least-squares (WTLS); Linearly structured.

Introduction

The total least-squares (TLS) method and its extensions have attracted widespread attention since the terminology of TLS was presented by Golub and Van Loan (1980). It is a popular approach for dealing with an overdetermined system of equations $\mathbf{Ax} \approx \mathbf{y}$ when both $\mathbf{A}$ and $\mathbf{y}$ are subject to errors, which is also known as an errors-in-variables (EIV) model. Essentially, the ordinary TLS (OTLS) does not depend on any prior stochastic assumption of the random variables. Nevertheless, it is proven to be a maximum likelihood estimator (MLE) when the errors of all elements in $[\mathbf{A} \mathbf{y}]$ are independent and identically distributed (i.i.d.) (Van Huffel and Vanderwalle 1991) from statistic standpoints. However, the i.i.d. assumptions are mostly unrealistic in applications. Therefore, extended methods have been proposed to overcome the disadvantage of the OTLS (Markovsky and Van Huffel 2007). Until recently, a large number of extended TLS methods have been proposed and widely applied in a variety of fields, such as signal processing, system identification, computer vision, geodesy, and so forth. New progress on EIV models with unknown variance components have been published by Xu and Liu (2013, 2014).

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Note. This manuscript was submitted on March 12, 2014; approved on February 18, 2016; published online on May 9, 2016. Discussion period open until October 9, 2016; separate discussions must be submitted for individual papers. This paper is part of the Journal of Surveying Engineering. © ASCE, ISSN 0733-9453.

The TLS and its extensions have been widely applied and have become an active topic in the geodetic community. The most frequently discussed extension of TLS in geodesy is weighted TLS (WTLS), in which the full statistical knowledge of the design matrix is taken into account. Quite a number of approaches for solving the WTLS problem have appeared in recent years. Schaffrin and Wieser (2008) developed a WTLS solution to linear regression, in which the cofactor matrix of design matrix $\mathbf{Q}_x$ is used for brevity) is restricted to a certain structure. Xu et al. (2012) extended the ordinary EIV model into a partial one, and presented a nonlinear TLS algorithm. Snow (2012) and Fang (2013) extended the WTLS algorithms in the case of cross-correlation between the design matrix and right-hand side (RHS) vector. Xu et al. (2014) presented a bias-corrected weighted least-squares (LS) method for a partial EIV model. Apart from the unconstrained cases, the constrained TLS (CTLS) or the WTLS solution has also been developed by a number of researchers (Mahboub and Sharifi 2013; Fang 2014a, b, 2015).

In contrast to the WTLS, the structured TLS (STLS) methods take advantage of the special structure of the design matrix when structured matrices are involved. The term STLS was first presented by De Moor (1993) in which $[\mathbf{A} \mathbf{y}]$ can be expanded as an affine matrix function of a noisy component vector and a group of fixed basis matrices. The structured matrices widely discussed are the Toeplitz or Hankel matrices (De Moor 1993; Lemmerling and Van Huffel 2001; Markovsky et al. 2005). A number of algorithms have been proposed to solve the STLS problems, in which CTLS (Abatzoglou et al. 1991), the Riemannian singular value decomposition (RiSVD) (De Moor 1993), and the structured total least norm (STLN) algorithms (Rosen et al. 1996; Van Huffel et al. 1996) are most frequently used. The extensions were also made to solve the nonlinear (Rosen et al. 1996; Lemmerling et al. 1996, 2002) or multivariate STLS problems (Kukush et al. 2005). In geodesy, the reformulation of a partial EIV model proposed by Xu et al. (2012) as a nonlinear adjustment problem automatically satisfies the requirement of structure preservation; thus it can be used to solve STLS problems. However, the STLS methods have not been as significantly investigated as the WTLS methods in geodetic literature.
Felus (2008) developed a STLS procedure for similarity transformation by following a Cadzow algorithm, in which a characteristic matrix mapping a random vector to the design matrix is introduced to preserve the structure. Felus (2006) performed a STLN algorithm for similar and affine coordinate transformation by treating them as STLS problems. However, these studies are restricted to homoscedastic cases and particular applications. Schaffrin et al. (2012) modified the Cadzow step to generate the optimal solution with the desired structure; however, it does not consider the cross-correlation between A and y. Fang (2014a) modified the WTLS algorithm by only considering the functionally independent errors to preserve the structure. Yet, the solution is based on converting the STLS problem into a WTLS framework.

The relationship between WTLS and STLS methods has also attracted the interests of geodetic researchers. In this paper, the classical STLS methods, including the CTLS algorithm and the STLN algorithm, are introduced but they are not familiar to geodesists yet. Based on these STLS methods, the quality assessment for the parameter estimates has been developed. The derivation presented in this paper on the unknown parameter accuracy is different from that by Xu et al. (2012). Xu et al. (2012) developed a STLS procedure for similarity transformation, such as line fitting, affine transformation and Bursa transformation, one can find that all the random elements in the design matrix consist of coordinates with errors. These error-contaminated coordinates can be equivalently treated as measurements with errors (Xu et al. 2012). In other words, the design matrix is the function of a measurement vector. In such cases, the well-known EIV model can be expressed as (Zhou et al. 2014)

\[ y \approx A(x) \xi \]  

where x is also called an input vector or variable. The errors of \( x \) are the functionally independent errors (Xu et al. 2012; Fang 2014a). Provided that \( x \) is a vector of size \( q \), \( A(x) \) is a function matrix that maps \( x \in R^q \) to \( A \in R^{m \times p} \). The elements of \( A \) may be constructed in a linear, quadratic, or even higher degree of \( x \) in applications. Here, the linear cases are restricted, such that

\[ A_{ij}(x) = z_{ij}^0 + z_{ij}^1 x; \forall i = 1 \cdots m; j = 1 \cdots p \]  

where \( A_{ij} \) is the element in the \( i \)th row and \( j \)th column of \( A \); \( z_{ij} \in R^q \) is a constant vector; and \( z_{ij}^1 \) is a fixed scalar. \( z_{ij} = 0 \) makes \( A_{ij} \) a fixed element; otherwise, it is random. The model \[ Eq. (3) \] with the constraint \[ Eq. (4) \] is a specific EIV model if the input and output vectors are corrupted by errors. Here, the concept linear-structured introduced by Beck and Ben-Tal (2005) is adopted and called a linear-structured EIV model. It is a special form of the partial EIV model proposed by Xu et al. (2012). The corresponding stochastic model of LSEIV is given by

\[ x = x_0 + e_x; y = y_0 + e_y \]  

\[ \begin{bmatrix} e_x \\ e_y \end{bmatrix} \sim N \left[ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} Q_x & Q_{xy} \\ Q_{yx} & Q_y \end{bmatrix} \right] \]  

where \( x_0, y_0 \) are the true values of \( x, y \); \( e_x, e_y \) are the additive error vectors of \( x_0, y_0 \); \( \sigma_0^2 \) is the covariance of unit weight; \( Q_x, Q_y \) are the cofactor matrices of \( x \) and \( y \); and \( Q_{xy} \) is the cross cofactor matrix of \( x \) and \( y \). It can be proven that the stochastic model \[ Eq. (6) \] can lead to \[ Eq. (2) \] in a LSEIV model.

Typically, the LSEIV model covers a wide range of geodetic applications. Here, only a few examples are given:

1. For data fitting (or regression), taking line fitting as an example, let there be \( m \geq 2 \) points, and the general LSEIV is given by

\[ A(x) = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix}; \ y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}; \ x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}; \ \xi = \begin{bmatrix} a \\ b \end{bmatrix} \]  

2. Geodetic transformation, such as two-dimensional (2D) similarity transformation (Neitzel 2010) or affine transformation (Felus 2004; Tong et al. 2011), and the Bursa datum transformation (Yang 1999; Grafarend and Awange 2003), in which the input vector consists of the coordinates in the source coordinate system and the output vector, are the corresponding coordinates in the target coordinate system. Taking the affine transformation as an example, assuming that there are \( n \) points altogether, it then yields
errors. If the corresponding to the nonrandom elements of input time series;

As described previously, quite a few WTLS algorithms (Schaffrin and Wieser 2008; Shen et al. 2011; Snow 2012; Xu et al. 2012; Fang 2013) are available for solving the EIV model. In a LSEIV model, the errors of [A y] are inherited from the measurement errors. If the coefficient matrix is structured, one must extract non-repetitive random elements (Xu et al. 2012). Xu et al. (2012) have proposed a partial EIV model by selecting functionally independent random elements within the coefficient matrix

\[
\mathbf{y} = (\mathbf{B}^T \otimes \mathbf{I}_n) (\mathbf{h} + \mathbf{B} \mathbf{a}) + \mathbf{e}_y
\]

\[
\mathbf{a} = \bar{\mathbf{a}} + \mathbf{e}_a
\]

where \( \mathbf{h} \) is a deterministic constant vector with elements corresponding to the nonrandom elements of \( \mathbf{A} \); and \( \mathbf{B} \) is a given deterministic matrix that represents the characteristic of matrix \( \mathbf{A} \). The \( m \)-dimensional vector \( \mathbf{a} \) collects all the independent random elements of \( \mathbf{A} \), and the true values of \( \mathbf{a} \) are denoted by \( \bar{\mathbf{a}} \). The weight matrix of \( \mathbf{a} \) is usually denoted by \( \mathbf{w} \).

In the partial EIV model, all the variables to be estimated can be written in the form of a compact vector \( \mathbf{b} = [\mathbf{B}^T; \bar{\mathbf{a}}^T] \). Compared with the existing EIV models, the partial EIV model proposed by Xu et al. (2012) is advantageous in dealing with the structured coefficient matrix and even generalizing the inequality and equality constraints for both the unknown parameter vector and the independent random elements within the coefficient matrix. For the solution of the partial EIV model, one can either regard the partial EIV model as a standard nonlinear Gauss-Markov model to find the standard nonlinear LS solution or use the TLS solution developed by Xu et al. (2012). Therefore the formulation of the solution will not be presented here again.

### STLS Formulation

In this section, the LSEIV model is formulated as a STLS problem and solved by the existing STLS algorithms. The STLS method was first introduced by De Moor (1993), in which \( \mathbf{A} \) is known to have a linear or affine structure. Ignoring the weight matrix, a STLS problem is given by (Markovsky et al. 2004)

\[
\min_{\mathbf{E}_A, \mathbf{e}_y, \xi} \| \mathbf{E}_A \mathbf{e}_y \|^2_F \quad \text{s.t.} \quad (\mathbf{A} - \mathbf{E}_A)\xi = (\mathbf{y} - \mathbf{e}_y) \]

\[
[\mathbf{E}_A \mathbf{e}_y] \text{ has the same structure as } [\mathbf{A} \mathbf{y}] \]

Examining the definition of Eqs. (1a), (1b), and (12a)–(12c), one can find that it is the constraint [Eq. (12c)] that distinguishes STLS from OTLS, allowing one to obtain an estimation of \( \xi \) when [\( \mathbf{A} \mathbf{y} \)] are structured. Furthermore, different weights may possibly be assigned to each element. Beck and Eldar (2010) gave a clear description of the STLS problem, in which the design matrix \( \mathbf{A} \) can be expressed as

\[
\mathbf{A} = \sum_{i=1}^{q} \alpha_i \mathbf{A}_i; \quad \alpha_i \in \mathbb{R}, i = 1, \cdots, q
\]

where \( \mathbf{A}_1, \ldots, \mathbf{A}_q \in \mathbb{R}^{m \times p} \) are the fixed structure (or basis) matrices; and \( \alpha_1, \ldots, \alpha_q \) are the structure components containing errors, and typically \( q \) is much smaller than \( m \times p \). Assuming \( \Delta \alpha_1, \ldots, \Delta \alpha_q \) are the unknown perturbations of the structure vector \( \alpha_1, \ldots, \alpha_q \), it yields

\[
\Delta \mathbf{A} = \sum_{i=1}^{q} \Delta \alpha_i \mathbf{A}_i, (i = 1, \cdots, q)
\]

where \( \Delta \mathbf{A} \) stands for the true errors of \( \mathbf{A} \). Eqs. (13) and (14) mean that \( \mathbf{A} \) and \( \Delta \mathbf{A} \) have the same structures. From the definition of STLS given earlier, an EIV model is a STLS problem only if the design matrix can be expanded as in Eq. (13). From the definition of Eqs. (3) and (4), the design matrix can be expressed as

\[
\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_q \]

where
It is clear that all the elements in $\mathbf{A}_i$ $(i = 0 \cdots q)$ are fixed values that consist of the basis matrices, and the input vector $\mathbf{x}$ is the noisy structured components. Moreover, $\mathbf{y}$ can also be verified to be a structured vector, although whether it is structured or not is not an important issue. Considering the errors in $\mathbf{x}$, it gives

$$
\mathbf{A}_i(\mathbf{x}_0 + \mathbf{e}_x) = \mathbf{z}_i^0 + \mathbf{z}_i^T \mathbf{x}_0 + \mathbf{e}_x = \mathbf{z}_i^0 + \mathbf{z}_i^T \mathbf{x}_0 + \sum_{k=1}^{q} \mathbf{z}_{ik}^T \mathbf{e}_k^i
$$

(16)

where $\mathbf{z}_{ik}^T$ and $\mathbf{e}_k^i$ are the $k$th element of the corresponding vectors. As a result, the error counterpart of the design matrix equals

$$
\mathbf{E}_A = \mathbf{e}_x \mathbf{A}_1 + \cdots + \mathbf{e}_x \mathbf{A}_q
$$

(17)

Eqs. (15) and (17) indicate that $\mathbf{A}$ and $\mathbf{E}_A$ have the same structures. The CTLS (Abatzoglou et al. 1991), RiSVD (De Moor 1993), and STLN algorithms (Rosen et al. 1996; Van Huffel et al. 1996) are the three most widely used approaches in signal procession literature. Xu et al. (2012) also developed a WTLS-like STLS algorithm, which can be regarded as a new formulation becomes

$$
\mathbf{T}_{p+1} = \frac{\partial \mathbf{y}}{\partial \mathbf{A}_i} = \left( \begin{array}{c}
\frac{\partial \mathbf{A}(1,1)}{\partial \mathbf{A}_i} & \cdots & \frac{\partial \mathbf{A}(1,i)}{\partial \mathbf{A}_i} \\
\vdots & \ddots & \vdots \\
\frac{\partial \mathbf{A}(m,1)}{\partial \mathbf{A}_i} & \cdots & \frac{\partial \mathbf{A}(m,i)}{\partial \mathbf{A}_i}
\end{array} \right) (i = 1 \cdots p)
$$

(19)

By introducing

$$
\mathbf{B} = [\mathbf{A} \ \mathbf{y}]; \ \mathbf{\eta} = (\mathbf{z}^T \ -1)^T; \ \mathbf{H} = -\sum_{i=1}^{p} \mathbf{z}_i \mathbf{T}_i \ \mathbf{T}_{p+1}
$$

(20)

Eq. (11b) is equal to

$$
\mathbf{He} + \mathbf{B}\mathbf{\eta} = 0
$$

(22)

Performing the Lagrange multiplier method from Eq. (18a) subject to Eq. (22), the CTLS problem can be changed into an unconstrained minimization problem as seen in Eq. (23) after some simple mathematic transformations (Abatzoglou et al. 1991)

$$
\min_{\xi} \mathbf{\eta}^T \mathbf{h}^T (\mathbf{H} \mathbf{Q} \mathbf{H}^T)^{-1} \mathbf{B}\mathbf{\eta}
$$

(23)

To solve Eq. (23), Abatzoglou et al. (1991) proposed a Newton’s method by using analytically derived gradients. Zhou and Pierre (2005) gave a simplified computation scheme of the algorithm. Some minor modifications are made in this paper to make the algorithm suitable for the LSEIV model. The computational procedures are outlined as:

1. Get $\mathbf{C}$ from the Cholesky factorization of the measurement covariance matrix, that is $\mathbf{C}^T = \mathbf{Q}_1$, let $\mathbf{G}_i = -\mathbf{T}_i \mathbf{C}^T (i = 1, \ldots , p + 1)$.
2. Use the OTLS solution as the initial value $\mathbf{\xi}_0$, give the convergence tolerance $\varepsilon$, and initialize the iteration counter $j = 0$.
3. Do the following calculation:

$$
\mathbf{H} = (\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{B}\mathbf{\eta}_j (\mathbf{\eta}_j = (\mathbf{z}^T \ -1)^T)
$$

$$
\mathbf{\eta}_j
$$

$$
\mathbf{A} = \mathbf{A} - [\mathbf{G}_1 \mathbf{H}_1 \mathbf{u}_1, \ldots , \mathbf{G}_p \mathbf{H}_p \mathbf{u}_1]; \ \mathbf{G} = [\mathbf{G}^1 \mathbf{u}_1, \ldots , \mathbf{G}^p \mathbf{u}_1]; \ \mathbf{v} = \mathbf{A}^T \mathbf{u}
$$

$$
\mathbf{M} = -\mathbf{G}^T \mathbf{H}(\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{A} - (\mathbf{G}^T \mathbf{H}(\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{A})^T
$$

$$
\mathbf{N} = [\mathbf{A}^T (\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{A}^T + (\mathbf{G}^T \mathbf{H}(\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{H} - \mathbf{I})]\mathbf{G}
$$

4. Update the parameter as

$$
\mathbf{\xi}_{(j+1)} = \mathbf{\xi}_{(j)} + (\mathbf{M} \mathbf{N}^{-1} \mathbf{M} - \mathbf{N}^{-1}) (\mathbf{v} - \mathbf{M} \mathbf{N}^{-1} \mathbf{v})
$$

5. Set $j = j + 1$, repeat Steps 3–4 until $\mathbf{\xi}$ converges.

It should be noted that the Newton algorithm cannot guarantee the convergent to the global optimal solution due to the nonconvex nature of the STLS problem. However, one still assumes it can correctly converge by assuming that the starting point is close enough.

The interested reader can refer to Abatzoglou et al. (1991) for a detailed description.
The STLN is another widely used STLS algorithm. It is formulated to minimize the error in either the $L_1$, $L_2$ or the $L$-norm (Rosen et al. 1998). Only the $L_2$ normal STLN problem is considered to keep the cost function consistent with the LS criteria. The STLN uses a vector $\alpha$ to represent the functionally independent errors of $E_A$ and $\beta$ to represent the functionally independent errors of $e$, that are not already contained in $\alpha$. The STLN is used to find the solution by minimizing the cost function of $\alpha^T P_\alpha \alpha + \beta^T P_\beta \beta$ (where $P_\alpha$ and $P_\beta$ are the weight matrices of $\alpha$ and $\beta$, respectively). It is obvious that $e$, and $e_i$ in a LSEIV model are $\alpha$ and $\beta$. Because the STLN algorithm is only taking that $\alpha$ and $\beta$ are uncorrelated into account, improvements are made for general consideration. The improved STLN formulation is given by

$$\min_{ \varepsilon } \frac{1}{2} e^T P e$$  \hfill (24a)

subject to:

$$A \varepsilon - G e = y - e $$  \hfill (24b)

with $G \in \mathbb{R}^{m \times d}$, $Ge = E_\xi \xi$; and the other notations are the same as those defined in Eqs. (18a) and (18b). The total residual vector $r$ is defined as

$$r(\hat{e}, \hat{\xi}) = -y + \hat{e} + A \hat{\xi} - G e + (A - E_A) \delta \xi$$  \hfill (25)

where $\hat{e}$ is the predicted value of the $e$, $\hat{\xi} = \xi + \delta \xi$, where $\xi$ is an approximated value close to $\hat{\xi}$; and $\delta \xi$ are the corresponding corrections. Considering $Ge = E_\xi \xi$, one gets

$$Ge = E_\xi \xi = (\xi^T \otimes I_m) e_A = (\xi^T \otimes I_m) J e$$  \hfill (26)

It leads to

$$G = (\xi^T \otimes I_m) J$$  \hfill (27)

A couple of algorithms have been proposed (Van Huffel et al. 1996; Rosen et al. 1996; 1998) to solve the STLN problem. They can be classified into two strategies: transforming the constrained optimization problem into an unconstrained one (Xu et al. 2012; Fang 2014c) and approximating the constraint [Eq. (24b)] in each step iteratively by a linear approximation while leaving the cost function unchanged. Here is a brief introduction to the maximal weight algorithm presented by Van Huffel et al. (1996), in which the constraint was unified into the cost function by assigning a maximal weight, which makes the constrained problem into an unconstrained one. Eqs. (24a) and (24b) are equal to

$$\min_{ \varepsilon } : \|r(\hat{e}, \hat{\xi})\|_F^2 = \|U \delta + r\|_F^2$$  \hfill (28)

with

$$U = \begin{pmatrix} \omega [G - I_m] & -\omega (A - E_A) \end{pmatrix}$$

where $\omega$ is the maximal weight of Eq. (24b); and $\| \cdot \|_F$ is the Frobenius norm. Eq. (27) is a traditional LS adjustment problem. It can be solved by an iterative method. The algorithm is summarized as

1. Choose a large weight $\omega$ and the convergence tolerance $\varepsilon$ and set $E_A = 0$, $e_C = e = 0$. Compute the initial value of $\xi$, and construct matrix $G$ through the equation $Ge = E_\xi \xi$, $r = y - A \hat{\xi}$

2. Repeat

   a. Compute the corrections $\delta = -(U^T U)^{-1} U^T r$

   b. Set $\hat{\xi} = \xi + \delta \xi = e + \delta e$

   c. Construct $E_A$ from $\hat{e}_C$ and $\hat{G}$ from $\hat{\xi}$, compute $r = y - e - (A - E_A) \hat{\xi}$ until $\| \delta \| < \varepsilon$

From the two STLS algorithms given earlier, one can find that their difference lies in how to expand the LSEIV model. The CTLS algorithm formulated the STLS problem by expanding [Ay] with multiplication of a noisy vector and the basis matrices from column to column and making STLS an unconstrained optimizing problem to be solved by a standard Gauss-Newton method. The STLN formulates the STLS problem by assigning a maximal weight to the model functions and treats the corrections of measurements and parameters as unknowns to be solved like an ordinary LS adjustment problem. The problem is that such a maximal weight may easily cause the overflow or truncated error in computation. Therefore, CTLS is preferable in solving a STLS problem.

Accuracy Assessment of STLS

The accuracy formula in Xu et al. (2012) is of the first-order approximation. The formula of the estimated accuracy and others also are of the same type, although these formulas might have different formulations. Because the STLS method actually is a nonlinear LS adjustment by treating the random vectors $x$ and $y$ as measurements, the variance component of unit weight gives

$$\hat{\sigma}_0^2 = \hat{e}^T P \hat{e} / m - p$$  \hfill (29)

Once the estimates are achieved, Eq. (22) can be rewritten as

$$H e + A (\hat{\xi} + \delta \hat{\xi}) - \hat{y} = 0$$  \hfill (30)

where $H = \left[ \sum_{i=1}^{p} T_i \right]^T A = A + [T_1 e_1; \ldots; T_p e_p]$; $\hat{y} = y + \hat{e}_C$; and $\delta \hat{\xi}$ is the perturbation vector of $\hat{\xi}$. Eq. (30) under the cost function of Eq. (18a) is a conditional adjustment with parameters. It gives

$$\delta \hat{\xi} = (A^T (H Q H^T)^{-1} A)^{-1} \left( A^T (H Q H^T)^{-1} (\hat{y} - A \hat{\xi}) \right)$$  \hfill (31)

With respect to $\hat{y} - A \hat{\xi} = \hat{H} e$, through error propagation law, one can get

$$Q_\xi = (A^T (H Q H^T)^{-1} A)^{-1}$$  \hfill (32)

As a result, Eqs. (29) and (32) provide a method for accuracy assessment of the CTLS method, and that of the STLN method can be established in a similar way.

To conclude this section, a simple comparison between the previously proposed methods is given. The STLS algorithm attempts to minimize the sum of the weighted squared errors of the noisy component vector. It is equal to the well-known LS criteria aimed at minimizing the sum of weighted measurement errors in geodesy. The WTLS seeks to minimize the sum of weighted squared errors of $A$ and $y$. It can be proven that the
proposed WTLS and STLS algorithms in a LSEIV model can be converted into the same unconstrained optimization problem (Abatzoglou et al. 1991; Zhou et al. 2014). The differences only arise from numerical computation.

It should be stressed that both WTLS and STLS are nonconvex optimization problems; consequently, the devised algorithms are not guaranteed to converge to a global optimum. Appropriate initial values or extra improvements are needed to guarantee the validity.

### Case Studies

Data experiments are performed in this section to examine the accuracy and efficiency performance of the proposed STLS algorithms for a LSEIV model, and comparisons are made with that of the WTLS methods as well.

#### Table 1. RMSE of Different Estimators

<table>
<thead>
<tr>
<th>Method</th>
<th>$a$</th>
<th>$B$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLS</td>
<td>0.005292410</td>
<td>0.0109221503</td>
<td>0.0526734212</td>
</tr>
<tr>
<td>WTLS</td>
<td>0.0053276355</td>
<td>0.0065547310</td>
<td>0.0376027566</td>
</tr>
<tr>
<td>CTLS-STLS</td>
<td>0.0053276355</td>
<td>0.0065547310</td>
<td>0.0376027566</td>
</tr>
<tr>
<td>STLN-STLS</td>
<td>0.0053276355</td>
<td>0.0065547310</td>
<td>0.0376027565</td>
</tr>
</tbody>
</table>

#### Fig. 1. Differences of parameters estimated by WTLS and STLS methods: (a) differences between the WTLS and CTLS algorithms; (b) differences between the WTLS and STLN algorithms
where

\[ \alpha, b, \text{and } c \text{ are the parameters of plane with true values of } a = 0.05, b = 0.2, \text{ and } c = 100. \]

The errors of different points are independent, but the errors within a point are correlated to simulate the correlations between \( \mathbf{A} \) and \( \mathbf{y} \). The errors of an arbitrary point \( \mathbf{r} \) are normally distributed with zero mean and covariance matrix given by

\[
\mathbf{V}_r = \begin{pmatrix}
\alpha^2 & \rho_1 \alpha \sigma_2 \sigma_3 & \rho_2 \sigma_2 \sigma_3 \\
\rho_1 \alpha \sigma_2 \sigma_3 & \sigma_2^2 & \rho_3 \sigma_2 \sigma_3 \\
\rho_2 \sigma_2 \sigma_3 & \rho_3 \sigma_2 \sigma_3 & \sigma_3^2
\end{pmatrix}
\]

where \( \mathbf{V}_r \) is a symmetric positive definite matrix with \( \sigma_2^2 = \sigma_3^2 = 0.5^2 \) and \( \sigma_1^2 = 0.1^2 \); and \( \rho_1, \rho_2, \text{ and } \rho_3 \) are the correlation coefficients randomly generated within \([-0.5, 0.5]\) at each simulation. OTLS and proposed WTLS, CTLS, and STLN algorithms are implemented for comparison purposes. The RMS error (RMSE) of each parameter defined by Eq. (35) from 1,000 trials is chosen as the criterion for accuracy comparison.

\[
\text{RMSE} = \pm \sqrt{\frac{\sum_{k=1}^{m} (\xi_k^0 - \xi_k^r)^2}{m}}
\]

where \( \xi_k^0 \) is the true value of the \( k \)th parameter in the \( k \)th trial; \( \xi_k^r \) is the estimated value of the \( k \)th parameter; and \( m \) is the total amount of simulations, which is \( m = 1,000 \).

Because the WTLS and STLS algorithms need iterative computations, all the iterations start from the solutions acquired by the OTLS method, and \( \epsilon = 10^{-10} \) is adopted by Schaffrin and Wieser (2008) and chosen as the convergence tolerance. In terms of the STLN algorithm, the maximum weight is set as \( \omega = 10^6 \). As the weight will be squared in the normal function formulation, it means the equivalent weight of the conditional function is at least \( 10^9 \) times that of the measurements. All the algorithms are performed with the software MATLAB 6.5.

The RMSE of previously mentioned algorithms are listed in Table 1, and all the results are kept 10 digits after decimals for comparison purposes. It shows that the RMSE of the WTLS and two STLS algorithms are almost equal except for the OTLS method. The differences of parameters estimated by different algorithms are almost equal except for the OTLS method. The differences of parameters estimated by different algorithms are almost equal except for the OTLS method.
methods in each trial are illustrated in Fig. 1. It shows the largest difference between WTLS and CTLS is less than 10^{-11}, which is within the given convergence tolerance. It can be concluded that the differences are mainly caused by numerical computation. The differences between STLN and WTLS are about a 10^{-8} order of magnitude, which is larger that that of CTLS. This is because the maximum weight assigned in computation is not the theoretical maximum, whereas the other two algorithms deteriorate the ill-condition of the normal matrix. Therefore, it is of vital importance in choosing an appropriate maximum weight to balance the accuracy and numerical stability in a STLN algorithm.

To compare the computational efficiency of each algorithm, the total amount of points varying from 100 to 1000 with a step of 100 is simulated. The average time cost of 100 trials at each quantity level is introduced for comparison. All the computations are performed by a Thinkpad X230 computer (Lenovo, Beijing), with Intel Core i5-3210M CPU and 2.5G RAM. The average time is shown in Fig. 2. The results indicate that the CTLS algorithm costs less time than those of the other two algorithms, and the STLN has the worst computational efficiency. The tendency implies the differences will increase significantly with the increasing amount of points because the corrections of measurements and parameters are solved simultaneously (which increase the order of normal matrix significantly) in a STLN algorithm, whereas the other two algorithms only solve the parameters in normal function and update the measurements afterward. The differences between CTLS and WTLS are not so significant because that they are mainly caused by the iterative formulas.

**Bursa Transformation**

Bursa model for 3D datum transformation is given by (Yang 1999):

\[
\begin{pmatrix}
X_i' \\
Y_i' \\
Z_i'
\end{pmatrix} = \begin{pmatrix}
X_i \\
Y_i \\
Z_i
\end{pmatrix} + \begin{pmatrix}
t_x \\
t_y \\
t_z
\end{pmatrix} + \mu \begin{pmatrix}
X_i' \\
Y_i' \\
Z_i'
\end{pmatrix}
+ \begin{pmatrix}
0 & \omega_z & -\omega_y \\
-\omega_z & 0 & \omega_x \\
\omega_y & -\omega_x & 0
\end{pmatrix} \begin{pmatrix}
X_i \\
Y_i \\
Z_i
\end{pmatrix} \tag{36}
\]

where \(t_x, t_y, \) and \(t_z\) are the three translations; \(\omega_x, \omega_y, \) and \(\omega_z\) are the three rotations and \(\mu\) is the scale parameter; and \((X_i', Y_i', Z_i')^T\) are the 3D coordinates of the source and target coordinates of the \(i\)th point, respectively. Let there be \(n (n \geq 3)\) point correspondences all together; then Eq. (36) can be rewritten in a general LSEIV model as

\[
y = \begin{pmatrix}
X_1' \\
Y_1' \\
Z_1'
\end{pmatrix} + \begin{pmatrix}
0 & 0 & -Y_1' \\
0 & 1 & Z_1' \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
X_1 \\
Y_1 \\
Z_1
\end{pmatrix} \tag{37}
\]

The coordinates of seven stations of the local and global reference systems (WGS 84) adopted from Grafarend and Awange (2003) are illustrated in Tables 2 and 3, respectively. Assuming that the three components of an arbitrary point are isotropic and independent, then \(\sigma^2_x = \sigma^2_y = \sigma^2_z = \sigma^2_{x,y} = \sigma^2_{x,z} = \sigma^2_{y,z} = 0.\) The corresponding errors of each point adopted from Grafarend and Awange (2003) are also shown in Tables 2 and 3.

From Eq. (37), \(A\) is a matrix of size \(3n \times 7,\) and the noisy component is \(\mathbf{1} = (X_i', Y_i', Z_i', \ldots, X_n', Y_n', Z_n'; X_1, Y_1, Z_1, \ldots, X_n, Y_n, Z_n)^T \in \mathbb{R}^7 (k = 6n).\) It is expanded in the Appendix to be solved by the proposed STLS algorithms. Because the translate parameters have a large quantity compared with the first example, the convergence tolerance \(\varepsilon = 10^{-6}\) is chosen. In terms of the STLN algorithm, the maximum weight is still \(\omega = 10^6.\) The results are listed in Table 4.

The three translates are kept in four digits and the other parameters are kept in 10 digits after the decimal. The results in Table 4 indicate that the difference between the three translates estimated by the mentioned algorithms are within \(\pm 0.2^\circ\)
mm, and the other parameters are completely the same.
Comparing with the nonlinear solutions of Grafarend and Awange (2003), it is found that the differences of three translates are within ±2 mm, and the differences of scale and rotation parameters can reach the 10⁻⁹ level. In terms of the adjustment results, one finds that the TLS is basically of no use for geodetic coordinate transformation unless one pursues high precise results at the millimeter level. This agrees with the viewpoint given in Xu et al. (2014), especially when the signal-to-noise ratio is extremely large. It shows that the solution of WTLS and STLS methods can achieve a high accuracy close to the nonlinear ones. The error of each parameter estimated by the proposed accuracy assessment method is also equal. As a result, the proposed WTLS and STLS methods can achieve the same accuracy. In terms of computational efficiency, the proposed WTLS algorithm needs more than 500 iteration times compared with three times for CTLS and five times for STLN. Such situations are from the severe ill-condition [Cond(A) = 2.77 x 10⁹] of the problem, except for the methodology differences described previously. In such cases, Tikhonov regularization (Tikhonov and Arsenin 1977), biased estimation (Hoerl and Kennard 1970) and/or truncated SVD (Xu 1998) methods can be implemented to overcome the ill-posedness.

Conclusions and Recommendations

The studies are restricted to a specific EIV model called the LSEIV model. The WTLS, CTLS, and STLN algorithms along with approximated accuracy assessment methods are introduced. Comparisons of accuracy and computational efficiency are made between the proposed STLS and WTLS methods. Through theoretical analysis and data experiments the conclusions maybe summarized as follows:

1. For a LSEIV model, the STLS can achieve the same accuracy as that of the WTLS method proposed in Xu et al. (2012). Furthermore, the formula of the variance factor is derived based on the STLS strategy.

2. To more fully investigate the potential of STLS adjustment methods for a range of geodetic applications, a number of areas need further studying, such as nonlinear structured EIV models, the ill-condition of STLS problems, the variance component estimation of STLS problems, and so forth.

Appendix. STLS Formulation of the Bursa Model

In the CTLS solution to the Bursa transformation parameters estimation, the design matrix A can be formulated as

\[
A_m = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
T_i = T_1 = T_2 = T_3 = 0; \quad T_{m+1} = \begin{pmatrix} U_i \\ m \times m \end{pmatrix} (i = 4, 5, 6, 7)
\]

where

\[
U_i = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
U_4 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

\[
U_5 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

In the STLS solution, the matrix G in Eq. (24b) is

\[
G = \begin{pmatrix}
\mu & \omega_z & -\omega_x \\
-\omega_z & \mu & \omega_y \\
\omega_x & -\omega_y & \mu \\
\vdots & \vdots & \vdots \\
0 & 0 & \cdots & \mu & \omega_x & -\omega_y \\
0 & 0 & \cdots & -\omega_x & \mu & \omega_y \\end{pmatrix}
\]

Acknowledgments

The authors thank the anonymous reviewers for their valuable comments and constructive suggestions on this manuscript. The work was partially supported by the National Natural Science Foundation of China (41274012, 41404005, and 41474006).

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