Quasi-Monte Carlo methods for optimal feedback control problems under uncertainty

Philipp Guth

with P. Kritzer (RICAM) and K. Kunisch (Uni Graz & RICAM)

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

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SS Optimization under uncertainty

The optimal control problem

Minimize $\mathcal{J}(y,u)$ subject to

$$\dot{y} = \mathcal{A}y + Bu + f, \qquad y(0) = y_\circ,$$

where

$$\mathcal{J}(y,u) = \frac{1}{2}\int_0^T \left(\|Q(y(t) - g(t))\|_H^2 + \|u(t)\|_U^2 \right) \mathrm{d}t + \frac{1}{2}\|P(y(T) - g_T)\|_H^2,$$

- V, H separable HS, Gelfand triplet $V \subset H = H' \subset V'$
- $y_\circ \in H$, $f \in L^2(0,T;V')$, $g_T \in H$, $g \in L^2(0,T;H)$ are given
- $B \in \mathcal{L}(U,H)$, with U finite-dimensional separable HS.
- $Q \in \mathcal{L}(H)$ and $P \in \mathcal{L}(H)$ observation operators
- $\mathcal{A} \in \mathcal{L}(V, V')$

The optimal control problem under uncertainty

Minimize $\mathcal{J}(y_{\boldsymbol{\sigma}}, u_{\boldsymbol{\sigma}})$ subject to

$$\dot{y}_{\boldsymbol{\sigma}} = \mathcal{A}_{\boldsymbol{\sigma}} y_{\boldsymbol{\sigma}} + B u_{\boldsymbol{\sigma}} + f, \qquad y_{\boldsymbol{\sigma}}(0) = y_\circ,$$

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- $B \in \mathcal{L}(U,H)$, with U finite-dimensional separable HS.
- $Q \in \mathcal{L}(H)$ and $P \in \mathcal{L}(H)$ observation operators
- $\mathcal{A}_{\boldsymbol{\sigma}} \in \mathcal{L}(V,V')$ depending on uncertain parameters $\boldsymbol{\sigma} \in \mathfrak{S}$

Robust open-loop problem

Find a deterministic control that is optimal under a risk measure $\mathcal{R}$.

Minimize $\mathcal{J}_{\mathcal{R}}(y_{\boldsymbol{\sigma}}, u)$ subject to

$$\dot{y}_{\boldsymbol{\sigma}} = \mathcal{A}_{\boldsymbol{\sigma}} y_{\boldsymbol{\sigma}} + Bu + f, \qquad y_{\boldsymbol{\sigma}}(0) = y_{\circ},$$

where

$$\mathcal{J}_{\mathcal{R}}(y_{\boldsymbol{\sigma}}, u) = \mathcal{R}\left(\frac{1}{2}\int_0^T \|Q(y_{\boldsymbol{\sigma}}(t) - g(t))\|_H^2 \,\mathrm{d}t + \frac{1}{2}\|P(y_{\boldsymbol{\sigma}}(T) - g_T)\|_H^2\right) + \int_0^T \|u(t)\|_U^2 \,\mathrm{d}t,$$

e.g., $\mathcal{R}(\cdot) = \int_{\mathfrak{S}} (\cdot)\, \mu(\mathrm{d}\boldsymbol{\sigma})$ or $\mathcal{R}(\cdot) = \theta^{-1} \ln\left(\int_{\mathfrak{S}} \mathrm{e}^{\theta\,(\cdot)}\, \mu(\mathrm{d}\boldsymbol{\sigma})\right)$, $\theta > 0$.

[GKKSS'24] Guth, P. A., Kaarnioja, V., Kuo, F. Y., Schillings, C., Sloan, I. H.: Parabolic PDE-constrained optimal control under uncertainty with entropic risk measure using quasi-Monte Carlo integration. *Numer. Math.* **156**, 565–608 (2024). https://doi.org/10.1007/s00211-024-01397-9

There is no feedback in open-loop problems

- Suppose that the robust OL-control u^*_{OL} has been computed
- Running the system with u^*_{OL}

$$\dot{y}_{\boldsymbol{\sigma}} = \mathcal{A}_{\boldsymbol{\sigma}} y_{\boldsymbol{\sigma}} + Bu^*_{\mathrm{OL}} + f, \qquad y_{\boldsymbol{\sigma}}(0) = y_\circ,$$

  leads after time T to $y_{\boldsymbol{\sigma}}(T; y_\circ; u^*_{\mathrm{OL}})$, $\boldsymbol{\sigma} \in \mathfrak{S}$.
- Suppose the "true" system state y_τ is available at $\tau \in (0, T)$
- To update the robust OL-control, one has to solve the optimization problem (with y_τ as initial condition) again!

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Find a robust control in feedback form: $u_{\boldsymbol{\sigma}} = K y_{\boldsymbol{\sigma}}$

How to find a good feedback law $K : H \to U$

Given $\boldsymbol{\sigma} \in \mathfrak{S}$, an optimal $u_{\boldsymbol{\sigma}}(t) = K_{\boldsymbol{\sigma}}(t, y_{\boldsymbol{\sigma}}(t))$ can be obtained, leading to the optimal closed-loop system

$$\dot{y}_{\boldsymbol{\sigma}}(t) = \mathcal{A}_{\boldsymbol{\sigma}} y_{\boldsymbol{\sigma}}(t) + BK_{\boldsymbol{\sigma}}(t, y_{\boldsymbol{\sigma}}(t)) + f(t), \qquad y_{\boldsymbol{\sigma}}(0) = y_{\circ}.$$

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Thus, we design $K = K_{\mathfrak{S}}$, based on the expectation w.r.t. $\boldsymbol{\sigma} \in \mathfrak{S}$:

$$K_{\mathfrak{S}} = \int_{\mathfrak{S}} K(\boldsymbol{\sigma})\, \mu(\mathrm{d}\boldsymbol{\sigma}) \approx \int_{\mathfrak{S}_s} K((\boldsymbol{\sigma}_s, \mathbf{0}))\, \mu_s(\mathrm{d}\boldsymbol{\sigma}_s) \approx \frac{1}{N} \sum_{k=0}^{N-1} K((\boldsymbol{\sigma}^{(k)}, \mathbf{0})),$$

and investigate QMC approximations of the integrals.

Optimality conditions

The optimal $(y(\boldsymbol{\sigma}), q_1(\boldsymbol{\sigma})) \in W_T(V, V') \times W_T(V, V')$ solves

$$G(\boldsymbol{\sigma}) \begin{pmatrix} y(\boldsymbol{\sigma}) \\ q_1(\boldsymbol{\sigma}) \end{pmatrix} = \begin{bmatrix} f \\ y_\circ \\ Q^* Q g \\ P^* P g_T \end{bmatrix} \quad \in V_T' \times H \times H_T \times H,$$

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where $G(\boldsymbol{\sigma}) \in \mathcal{L}(W_T(V, V') \times W_T(V, V'), V_T' \times H \times V_T' \times H)$ is

$$G(\boldsymbol{\sigma}) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{dt}} - \mathcal{A}_{\boldsymbol{\sigma}} & -BB^* \\ E_0 & 0 \\ Q^* Q & -\frac{\mathrm{d}}{\mathrm{dt}} - \mathcal{A}_{\boldsymbol{\sigma}}^* \\ P^* P E_T & E_T \end{bmatrix},$$

where $E_t : W_T(V, V') \to H$ for $t \in [0, T]$.

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where $E_t : W_T(V, V') \to H$ for $t \in [0, T]$. The optimal control is

$$u(\boldsymbol{\sigma}) = B^* q_1(\boldsymbol{\sigma}).$$

Parametric regularity of bounded linear operators

A fam. $\{\mathbb{G}(\boldsymbol{\sigma}) \in \mathcal{L}(X, Y') : \boldsymbol{\sigma} \in \mathfrak{S}\}$ is *p-analytic* for some $0 < p \leq 1$, if

(i) The operator $\mathbb{G}(\boldsymbol{\sigma})$ is invertible for every $\boldsymbol{\sigma} \in \mathfrak{S}$ with

$$\sup_{\boldsymbol{\sigma} \in \mathfrak{S}} \|\mathbb{G}(\boldsymbol{\sigma})^{-1}\|_{\mathcal{L}(Y',X)} \leq C\,.$$

(ii) For each $\boldsymbol{\sigma} \in \mathfrak{S}$, $\mathbb{G}(\boldsymbol{\sigma})$ is a real analytic function w.r.t. $\boldsymbol{\sigma}$. I.e. $\exists$ a nonnegative sequence $\tilde{\boldsymbol{b}} = (\tilde{b}_j)_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ such that for all $\boldsymbol{\nu} \in \mathcal{F} \setminus \{0\}$ there holds that

$$\sup_{\boldsymbol{\sigma} \in \mathfrak{S}} \|\mathbb{G}(\boldsymbol{0})^{-1}\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\mathbb{G}(\boldsymbol{\sigma})\|_{\mathcal{L}(X)} \leq C\tilde{\boldsymbol{b}}^{\boldsymbol{\nu}}$$

$$\mathcal{F} := \{\boldsymbol{m} \in \mathbb{N}_0^{\mathbb{N}} \mid \textstyle\sum_{j\geq 1} m_j < \infty\}$$

Example: Affine parameter dependence

$$\mathbb{G}(\boldsymbol{\sigma}) = \mathbb{G}_0 + \sum_{j\geq 1} \sigma_j \mathbb{G}_j$$

- If $\sup_{\boldsymbol{\sigma}\in\mathfrak{S}} \|\mathbb{G}_0^{-1}\|_{\mathcal{L}(Y',X)} \leq C_0$ and $\sum_{j\geq 1} \|\mathbb{G}_0^{-1}\mathbb{G}_j\|_{\mathcal{L}(X)} \leq \kappa < 2$, then $\mathbb{G}(\boldsymbol{\sigma})$ satisfies (i) and (ii) with $C = ((1-\frac{\kappa}{2})C_0^{-1})^{-1}$ and $\tilde{b}_j = \|\mathbb{G}_0^{-1}\mathbb{G}_j\|_{\mathcal{L}(X,Y')}$
- For every $f \in Y'$, $\exists! y(\boldsymbol{\sigma})$ such that $\mathbb{G}(\boldsymbol{\sigma})y(\boldsymbol{\sigma}) = f$. The parametric family $y(\boldsymbol{\sigma})$ depends analytically on the $\boldsymbol{\sigma} \in \mathfrak{S}$ with

$$\sup_{\boldsymbol{\sigma}\in\mathfrak{S}} \|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}} y(\boldsymbol{\sigma})\| \leq C\|f\|_{Y'}|\boldsymbol{\nu}|!\boldsymbol{b}^{\boldsymbol{\nu}}$$

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Lemma (Uniform boundedness of G_σ^{-1})

Let $\mathcal{A}_\sigma$ be associated with a uniformly V-H-coercive bilinear form and uniformly bounded. The family of operators $G_\sigma = G(\sigma) \in \mathcal{L}(W_T(V,V') \times W_T(V,V'), V_T' \times H \times V_T' \times H)$ has uniformly bounded inverses

$$\|G_\sigma^{-1}\|_{\mathcal{L}(V_T' \times H \times V_T' \times H, W_T(V,V') \times W_T(V,V'))} \leq c_{\mathcal{G}}(T), \quad \forall \sigma \in \mathfrak{S},$$

with $T \mapsto \tilde{c}_{\mathcal{G}}(T)$ cont. and monot. incr. and independent of σ.

Similar results in [KS'13], using $\sup_{t\in[0,T]} \|v\|_H \leq \varrho(T)\|v\|_{W_T(V,V')}$. But, $\varrho(T) \to \infty$ as $T \to \infty$, since $\|v\|^2_{W_T(V,V')} = T\|v\|^2_V = T\frac{\|v\|^2_V}{\|v\|^2_H}\|v\|^2_H$ for constant functions $v \in W_T(V,V')$, thus $\varrho(T) \geq \frac{1}{\sqrt{T}}\frac{\|v\|^2_H}{\|v\|^2_V}$.

[KS'13] Kunoth, A., Schwab, Ch.: Analytic Regularity and GPC Approximation for Control Problems Constrained by Linear Parametric Elliptic and Parabolic PDEs. *SIAM J. Control. Optim.* **51** (2013), pp. 2442–2471. https://doi.org/10.1137/110847597

Theorem ($G_{\boldsymbol{\sigma}}$ is p-analytic)

Let $\mathcal{A}_{\boldsymbol{\sigma}}$ be p-analytic with $c_{\mathcal{A}} > 0$ and $\tilde{\boldsymbol{b}} \in \ell^p(\mathbb{N})$, for some $0 < p \leq 1$. Then, for every $\boldsymbol{\sigma} \in \mathfrak{S}$, the FOC of the tracking problem can be formulated as an operator equation, and the associated operator $G_{\boldsymbol{\sigma}}$ is p-analytic with the same p. Moreover, the optimal state-adjoint-pair depends analytically on $\boldsymbol{\sigma} \in \mathfrak{S}$:

$$\left\| \partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}} \begin{pmatrix} y \\ q_1 \end{pmatrix} (\boldsymbol{\sigma}) \right\|_{W_T(V,V') \times W_T(V,V')} \leq \tilde{c}_{\mathcal{G}}(T) |\boldsymbol{\nu}|! \boldsymbol{b}^{\boldsymbol{\nu}} \left\| \begin{pmatrix} f \\ y_\circ \\ Q^* Q g \\ P^* P g_T \end{pmatrix} \right\|_{V_T' \times H \times V_T' \times H}$$

for all $\boldsymbol{\nu} \in \mathcal{F}$, with $b_j := \tilde{b}_j / \ln 2$, and a constant $\tilde{c}_{\mathcal{G}}(T) > 0$ depending cont. and monot. incr. on T and independent of $\boldsymbol{\sigma} \in \mathfrak{S}$.

Lemma (Regularity of the optimal cost)

There holds

$$\begin{aligned}
\left|\partial_{\boldsymbol{\sigma}}^{\boldsymbol{\nu}} \|Q(y(\boldsymbol{\sigma}) - g)\|_{H_T}^2\right| &\leq C_1(T)(|\boldsymbol{\nu}| + 1)!\, \boldsymbol{b}^{\boldsymbol{\nu}} \\
\left|\partial_{\boldsymbol{\sigma}}^{\boldsymbol{\nu}} \|u(\boldsymbol{\sigma})\|_{U_T}^2\right| &\leq C_2(T)(|\boldsymbol{\nu}| + 1)!\, \boldsymbol{b}^{\boldsymbol{\nu}} \\
\left|\partial_{\boldsymbol{\sigma}}^{\boldsymbol{\nu}} \|P(y_T(\boldsymbol{\sigma}) - g_T(\boldsymbol{\sigma}))\|_{H}^2\right| &\leq C_3(T)(|\boldsymbol{\nu}| + 1)!\, \boldsymbol{b}^{\boldsymbol{\nu}},
\end{aligned}$$

for all $\boldsymbol{\nu} \in \mathcal{F}$. *In particular, we have for the optimal cost*

$$\left|\partial_{\boldsymbol{\sigma}}^{\boldsymbol{\nu}} \mathcal{J}(y_{\boldsymbol{\sigma}}, u_{\boldsymbol{\sigma}})\right| \leq \frac{C_4(T)}{2}(|\boldsymbol{\nu}| + 1)!\boldsymbol{b}^{\boldsymbol{\nu}},$$

for all $\boldsymbol{\nu} \in \mathcal{F}$ *with* $C_4(T) = \sum_{i=1}^3 C_i(T)$ *depending cont. and monot. incr. on* T.

Theorem (Feeback regularity – homogeneous case)

The optimal feedback $K_{\boldsymbol{\sigma}}(t) = -B^*\Pi_{\boldsymbol{\sigma}}(T-t)$ *depends analytically on* $\boldsymbol{\sigma} \in \mathfrak{S}$, *and* $\forall t \in [0, T]$, *and* $\forall \boldsymbol{\nu} \in \mathcal{F}$ *we have*

$$\|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}(-B^*\Pi_{\boldsymbol{\sigma}}(T-t))\|_{\mathcal{L}(H,U)} \leq \|B\|_{\mathcal{L}(U,H)} C_5(T)(|\boldsymbol{\nu}|+1)!\boldsymbol{b}^{\boldsymbol{\nu}}.$$

Proof.

- $\dot{\Pi}_{\boldsymbol{\sigma}} = \Pi_{\boldsymbol{\sigma}}\mathcal{A}_{\boldsymbol{\sigma}} + \mathcal{A}^*_{\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}} - \Pi_{\boldsymbol{\sigma}}BB^*\Pi_{\boldsymbol{\sigma}} + Q^*Q, \quad \Pi_{\boldsymbol{\sigma}}(0) = P^*P.$
- $\frac{1}{2}\langle \Pi_{\boldsymbol{\sigma}}(T)y_\circ, y_\circ\rangle_H = \mathcal{J}(y_{\boldsymbol{\sigma}}, u_{\boldsymbol{\sigma}})$ and the prev. Lemma
- $\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\langle \Pi_{\boldsymbol{\sigma}}(T)y_\circ, y_\circ\rangle_H = \langle \partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}}(T)y_\circ, y_\circ\rangle_H$ for all $\boldsymbol{\nu} \in \mathcal{F}$ from the regularity of the adjoint $q_{1,\boldsymbol{\sigma}}(0) = \Pi_{\boldsymbol{\sigma}}(T)y_\circ$
- $\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}}(T)$ is bounded, linear and self-adjoint, thus $\|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}}(T)\|_{\mathcal{L}(H)} = \sup_{\|y_\circ\|_H=1} |\langle \partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}\Pi_{\boldsymbol{\sigma}}(T)y_\circ, y_\circ\rangle_H$.
- Autonomy of $\mathcal{A}_{\boldsymbol{\sigma}}, B, P$, and Q gives $\Pi_{\boldsymbol{\sigma}}(\tau)$ for $\tau \in [0, T]$ is restriction to $[0, \tau]$

We assume $f \in H_T$ and $g \in W^{1,2}(0,T;H) \cap L^2(0,T;D(\mathcal{A}_\sigma))$ and set $x_\sigma := y_\sigma - g$ and $r_\sigma := f + \mathcal{A}_\sigma g(t) - \dot{g}(t)$.

Theorem (Affine feedback – nonhomogeneous case)

For every $\sigma \in \mathfrak{S}$ *there exists a unique minimizer* (x_σ, u_σ) *satisfying, for* $t \in (0,T)$,

$$u_\sigma(t) = -B^*\big(\Pi_\sigma(T-t)x_\sigma(t) + h_\sigma(t)\big)$$
$$\dot{x}_\sigma(t) = \left(\mathcal{A}_\sigma - BB^*\Pi_\sigma(T-t)\right)x_\sigma(t) - BB^*h_\sigma(t) + r_\sigma(t)$$

with $x(0) = x_\circ$, *where*

$$-\dot{h}_\sigma(t) = \left(\mathcal{A}_\sigma^* - \Pi_\sigma(T-t)BB^*\right)h_\sigma(t) + \Pi_\sigma(T-t)r_\sigma(t)$$

with $h_\sigma(T) = 0$.

Proposition (Regularity of $h_{\boldsymbol{\sigma}}$)

Let $D(\mathcal{A}_{\boldsymbol{\sigma}})$ *be independent of* $\boldsymbol{\sigma} \in \mathfrak{S}$ *and* $D(\mathcal{A}_{\boldsymbol{\sigma}}) = D(\mathcal{A}_{\boldsymbol{\sigma}}^*)$ *for all* $\boldsymbol{\sigma} \in \mathfrak{S}$. *Further, let* $\|\mathcal{A}_{\boldsymbol{\sigma}}\|_{\mathcal{L}(D(\mathcal{A}),H))} \leq \widetilde{C}_{\mathcal{A}}$ *for all* $\boldsymbol{\sigma} \in \mathfrak{S}$. *Then*

$$\|\partial_{\boldsymbol{\sigma}}^{\boldsymbol{\nu}} h_{\boldsymbol{\sigma}}\|_{W_T^0(V,V')} \leq \Theta_{|\boldsymbol{\nu}|}(|\boldsymbol{\nu}|+2)!\boldsymbol{b}^{\boldsymbol{\nu}}, \quad \forall \boldsymbol{\nu} \in \mathcal{F},$$

where $\Theta_{|\nu|} = \frac{1}{2}(1+C)^{\max\{|\boldsymbol{\nu}|-1,0\}} C^{\delta_{\nu,\mathbf{0}}}(C+C^2)^{1-\delta_{|\nu|,0}}$, *with some constant* $C > 0$ *independent of* $\boldsymbol{\sigma} \in \mathfrak{S}$.

Proof.

With $D_{\boldsymbol{\sigma}} : W_T^0(V,V') \to V_T'$
as $D_{\boldsymbol{\sigma}} := -\frac{\mathrm{d}}{\mathrm{d}t} - (\mathcal{A}_{\boldsymbol{\sigma}}^* - \Pi_{\boldsymbol{\sigma}}(T-t)BB^*)$, we can write
$D_{\boldsymbol{\sigma}} h_{\boldsymbol{\sigma}} = \Pi_{\boldsymbol{\sigma}} r_{\boldsymbol{\sigma}}$. Using all prev. regularity results, the result is shown by induction on $|\boldsymbol{\nu}|$. □

Regularity of the feedback $K_{\boldsymbol{\sigma}}$

With $\max_{t\in[0,T]} \|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}} h_{\boldsymbol{\sigma}}(t)\|_H \leq C(T)\|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}} h_{\boldsymbol{\sigma}}\|_{W_T(V,V')}$ we can summarize our regaularity analysis:

Theorem (Combined regularity result)

In both, the homogeneous and the nonhomogeneous case, we have

$$\sup_{t\in[0,T]} \left(\|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}(-B^*\Pi_{\boldsymbol{\sigma}}(T-t))\|_{\mathcal{L}(H,U)} + \|\partial^{\boldsymbol{\nu}}_{\boldsymbol{\sigma}}(-B^* h_{\boldsymbol{\sigma}}(t))\|_U\right) \leq \|B\|_{\mathcal{L}(H,U)} C(T)(|\boldsymbol{\nu}|+2)!\boldsymbol{b}^{\boldsymbol{\nu}},$$

for all $\boldsymbol{\nu} \in \mathcal{F}$ *and a constant* $C(T) > 0$ *independent of* $\boldsymbol{\sigma} \in \mathfrak{S}$.

We are ready for the error analyisis of the feedback operator!

Dimension truncation

For $t \in [0,T]$, we are interested in integrals of the form

$$\int_{[-\frac{1}{2},\frac{1}{2}]^{\mathbb{N}}} \mathscr{K}_{\boldsymbol{\sigma}}(t)\,\mathrm{d}\boldsymbol{\sigma}, \qquad \text{where } \mathrm{d}\boldsymbol{\sigma} = \bigotimes_{j=1}^{\infty} \mathrm{d}\sigma_j \text{ and}$$

$$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^*\Pi_{\boldsymbol{\sigma}}(T-t) \in Z = \mathcal{L}(H,U), \quad \text{or} \quad \mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^*h_{\boldsymbol{\sigma}}(t) \in Z = U.$$

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Assuming $\|\mathscr{K}_{\boldsymbol{\sigma}}(t) - \mathscr{K}_{\boldsymbol{\sigma},s}(t)\|_Z \xrightarrow{s\to 0} 0$, [GK'24] yields

$$\left\| \int_{[-\frac{1}{2},\frac{1}{2}]^{\mathbb{N}}} (\mathscr{K}_{\boldsymbol{\sigma}}(t) - \mathscr{K}_{\boldsymbol{\sigma},s}(t))\,\mathrm{d}\boldsymbol{\sigma} \right\|_Z \leq C\, s^{-\frac{2}{p}+1},$$

where $C > 0$ is independent of s.

[GK'24] Guth, P. A., Kaarnioja, V.: Generalized Dimension Truncation Error Analysis for High-Dimensional Numerical Integration: Lognormal Setting and Beyond. *SIAM J. Numer. Anal.* **62** (2014), pp. 872–892. https://doi.org/10.1137/23M1593188

Dimension truncation

For $t \in [0, T]$, we are interested in integrals of the form

$$\int_{[-\frac{1}{2},\frac{1}{2}]^{\mathbb{N}}} \mathscr{K}_{\boldsymbol{\sigma}}(t)\,\mathrm{d}\boldsymbol{\sigma}, \qquad \text{where } \mathrm{d}\boldsymbol{\sigma} = \bigotimes_{j=1}^{\infty} \mathrm{d}\sigma_j \text{ and}$$

$$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^*\Pi_{\boldsymbol{\sigma}}(T-t) \in Z = \mathcal{L}(H,U), \quad \text{or} \quad \mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^* h_{\boldsymbol{\sigma}}(t) \in Z = U.$$

In the following we develop QMC rules to approximate

$$\int_{[-\frac{1}{2},\frac{1}{2}]^{s}} \mathscr{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma}_s,$$

where $\mathscr{K}_{\boldsymbol{\sigma},s} := \mathscr{K}_{\boldsymbol{\sigma}}((\sigma_1,\ldots,\sigma_s,0,0,\ldots))$, and $\mathrm{d}\boldsymbol{\sigma}_s = \bigotimes_{j=1}^{s} \mathrm{d}\sigma_j$.

[GK'24] Guth, P. A., Kaarnioja, V.: Generalized Dimension Truncation Error Analysis for High-Dimensional Numerical Integration: Lognormal Setting and Beyond. *SIAM J. Numer. Anal.* **62** (2014), pp. 872–892. https://doi.org/10.1137/23M1593188

Theorem ([GKKSS'24])

Let $\mathcal{W}_s$ *be a BS of functions* $F : [-\frac{1}{2}, \frac{1}{2}]^s \to \mathbb{R}$. *Consider an* N*-point QMC rule with integration nodes* $\boldsymbol{\sigma}^{(0)}, \ldots, \boldsymbol{\sigma}^{(N-1)} \in [-\frac{1}{2}, \frac{1}{2}]^s$, *given by* $Q_{N,s}(F) := \frac{1}{N}\sum_{k=0}^{N-1} F(\boldsymbol{\sigma}^{(k)})$. *Furthermore, we define the worst case error of integration using* $Q_{N,s}$ *in* $\mathcal{W}_s$ *by*

$$e^{\mathrm{wor}}(Q_{N,s}, \mathcal{W}_s) := \sup_{\substack{F \in \mathcal{W}_s \\ \|F\|_{\mathcal{W}_s} \le 1}} \left| \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^s} F(\boldsymbol{\sigma}) \, \mathrm{d}\boldsymbol{\sigma} - Q_{N,s}(F) \right|.$$

For a separable BS Z *and a continuous mapping* $k : [-\frac{1}{2}, \frac{1}{2}]^s \to Z$ *we have*

$$\left\| \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^s} k(\boldsymbol{\sigma}) \, \mathrm{d}\boldsymbol{\sigma} - \frac{1}{N} \sum_{k=0}^{N-1} k(\boldsymbol{\sigma}^{(k)}) \right\|_Z \le e^{\mathrm{wor}}(Q_{N,s}, \mathcal{W}_s) \sup_{\substack{G \in Z' \\ \|G\|_{Z'} \le 1}} \|G(k)\|_{\mathcal{W}_s}.$$

For the feedback choose $k = \mathscr{K}_{\sigma,s}(t)$, and $Z \in \{U, \mathcal{L}(H, U)\}$.

Randomly shifted rank-1 lattice rules

When choosing

$$\|F\|^2_{\mathcal{V}_{s,1,\boldsymbol{\gamma}}} := \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}^2} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{|\mathfrak{u}|}} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{\sigma}_{\mathfrak{u}}} F(\boldsymbol{\sigma}_{\mathfrak{u}};\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}})\,\mathrm{d}\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{\sigma}_{\mathfrak{u}}.$$

a RSLR can be constructed using a CBC algorithm such that

$$\mathbb{E}_{\boldsymbol{\Delta}}\left(\left\| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} \mathscr{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma} - \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}_{\Delta}^{(k)},s}(t) \right\|_Z^2\right) \le C_{s,1,\boldsymbol{\gamma},\lambda} \frac{1}{(\phi_{\mathrm{tot}}(N))^{1/\lambda}},$$

for all $\lambda \in (\frac{1}{2},1]$, where $\boldsymbol{\sigma}_{\Delta}^{(k)} = \boldsymbol{\sigma}^{(k)} + \Delta$, with $\Delta \sim \mathcal{U}([0,1]^s)$, and

$$C_{s,1,\boldsymbol{\gamma},\lambda} = \widetilde{C}^2 \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{2\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}\right)^{|\mathfrak{u}|} \right)^{\frac{1}{\lambda}} \left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{[(|\mathfrak{u}|+2)!]^2 \prod_{j\in\mathfrak{u}} b_j^2}{\gamma_{\mathfrak{u}}^2} \right),$$

with $\widetilde{C} := \|B\|_{\mathcal{L}(U,H)} C(T)$.

Randomly shifted rank-1 lattice rules

When choosing

$$\|F\|^2_{\mathcal{V}_{s,1,\gamma}} := \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}^2} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{|\mathfrak{u}|}} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{\sigma}_{\mathfrak{u}}} F(\boldsymbol{\sigma}_{\mathfrak{u}};\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}})\mathrm{d}\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{\sigma}_{\mathfrak{u}}.$$

a RSLR can be constructed using a CBC algorithm such that

$$\mathbb{E}_{\boldsymbol{\Delta}}\left(\left\| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} \mathscr{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma} - \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}_{\boldsymbol{\Delta}}^{(k)},s}(t) \right\|_Z^2 \right) \le C_{s,1,\gamma,\lambda}\,\frac{1}{(\phi_{\mathrm{tot}}(N))^{1/\lambda}},$$

for all $\lambda \in (\frac{1}{2},1]$, and where $C_{s,1,\gamma^*,\lambda}$ is bounded independently of s, for

$$\gamma^* = (\gamma_{\mathfrak{u}}^*)_{\mathfrak{u}\subseteq\{1,\ldots,s\}} = \left((|\mathfrak{u}|+2)! \prod_{j\in\mathfrak{u}} \frac{b_j(2\pi^2)^{\lambda/2}}{\sqrt{2\zeta(2\lambda)}} \right)^{1/(1+\lambda)},$$

and

$$\lambda = \begin{cases} \frac{1}{2-2\delta} & \text{for arbitrary } \delta\in(0,1) \quad \text{if } p\in(0,\frac{2}{3}], \\ \frac{p}{2-p} & \text{if } p\in(\frac{2}{3},1]. \end{cases}$$

Randomly shifted rank-1 lattice rules

When choosing

$$\|F\|^2_{\mathcal{V}_{s,1,\gamma}} := \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}^2} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{|\mathfrak{u}|}} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{\sigma}_{\mathfrak{u}}} F(\boldsymbol{\sigma}_{\mathfrak{u}}; \boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}}) \mathrm{d}\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{\sigma}_{\mathfrak{u}}.$$

a RSLR can be constructed using a CBC algorithm such that

$$\mathbb{E}_{\boldsymbol{\Delta}} \left(\left\| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} \mathscr{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma} - \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}_{\Delta}^{(k)},s}(t) \right\|_Z^2 \right) \leq C_{s,1,\gamma,\lambda} \frac{1}{(\phi_{\text{tot}}(N))^{1/\lambda}},$$

for all $\lambda \in (\frac{1}{2}, 1]$. Consequently, the MSE is of order

$$\kappa(N) = \begin{cases} [\phi_{\text{tot}}(N)]^{-2-2\delta} & \text{for arbitrary } \delta \in (0,1) \quad \text{if } p \in (0, \frac{2}{3}], \\ [\phi_{\text{tot}}(N)]^{-\left(\frac{2}{p}-1\right)} & \text{if } p \in (\frac{2}{3}, 1]. \end{cases}$$

Interlaced polynomial lattice rules

When choosing b prime, $m \in \mathbb{N}$ and $N = b^m$, as well as

$$\|F\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,\infty}} \max_{\mathfrak{u}\subseteq\{1,\ldots,s\}} \Bigg(\frac{1}{\gamma_{\mathfrak{u}}^q} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1,\ldots,\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}}$$

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{|\mathfrak{v}|}} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{s-|\mathfrak{v}|}} \frac{\partial^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\mathbf{0})}}{\partial\boldsymbol{\sigma}_{\mathfrak{u}}} F(\boldsymbol{\sigma}_{\mathfrak{u}};\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}}) \mathrm{d}\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{v}} \right|^q \mathrm{d}\boldsymbol{\sigma}_{\mathfrak{v}} \Bigg)^{\frac{1}{q}}$$

an ILPR of order α can be constructed using a CBC algorithm such that

$$\left\| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} \mathcal{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma} - \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{K}_{\boldsymbol{\sigma}^{(k)},s}(t) \right\|_Z \le C_{s,\alpha,\boldsymbol{\gamma},\lambda} \frac{1}{(N-1)^{1/\lambda}}$$

for all $\lambda \in (\frac{1}{\alpha}, 1]$, where $\rho_{\alpha,b}(\lambda) := \left(C_{\alpha,b}\, b^{\alpha(\alpha-1)/2}\right)^\lambda \left(\left(1+\frac{b-1}{b^{\alpha\lambda}-b}\right)^\alpha - 1\right)$,

$$C_{s,\alpha,\boldsymbol{\gamma},\lambda} = \widetilde{C}\Bigg(2 \sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^\lambda (\rho_{\alpha,b}(\lambda))^{|\mathfrak{u}|}\Bigg)^{\frac{1}{\lambda}} \frac{1}{\gamma_{\mathfrak{u}}} \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} (|\boldsymbol{\nu}_{\mathfrak{u}}|+2)! \prod_{j\in\mathfrak{u}} (2^{\delta(\nu_j,\alpha)} b_j^{\nu_j}).$$

Interlaced polynomial lattice rules

When choosing b prime, $m \in \mathbb{N}$ and $N = b^m$, as well as

$$\|F\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,\infty}} \max_{\mathfrak{u}\subseteq\{1,\dots,s\}} \left(\frac{1}{\gamma_{\mathfrak{u}}^q} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1,\dots,\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{|\mathfrak{v}|}} \left| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^{s-|\mathfrak{v}|}} \frac{\partial^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\mathbf{0})}}{\partial\boldsymbol{\sigma}_{\mathfrak{u}}} F(\boldsymbol{\sigma}_{\mathfrak{u}};\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{u}})\mathrm{d}\boldsymbol{\sigma}_{\{1:s\}\setminus\mathfrak{v}} \right|^q \mathrm{d}\boldsymbol{\sigma}_{\mathfrak{v}} \right)^{\frac{1}{q}}$$

an ILPR of order α can be constructed using a CBC algorithm such that

$$\left\| \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^s} \mathscr{K}_{\boldsymbol{\sigma},s}(t)\,\mathrm{d}\boldsymbol{\sigma} - \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}^{(k)},s}(t) \right\|_Z \leq C_{s,\alpha,\boldsymbol{\gamma},\lambda} \frac{1}{(N-1)^{1/\lambda}}$$

for all $\lambda \in (\frac{1}{\alpha}, 1]$ and where $C_{s,\alpha,\boldsymbol{\gamma},\lambda}$ is bounded independently of s, for

$$\gamma_{\mathfrak{u}}^* := \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} (|\boldsymbol{\nu}_{\mathfrak{u}}|+2)! \prod_{j\in\mathfrak{u}} (2^{\delta(\nu_j,\alpha)} b_j^{\nu_j}),$$

and $\lambda = p$ and $\alpha = \lfloor \frac{1}{p} \rfloor + 1$. Thus the integration error is of order $\mathcal{O}(N^{-\alpha})$.

Consider the parameterized convection-diffusion-reaction equation

$$
\begin{aligned}
\dot{y}_{\boldsymbol{\sigma}} - \nabla \cdot (a_{\boldsymbol{\sigma}} \nabla y_{\boldsymbol{\sigma}}) + c y_{\boldsymbol{\sigma}} + \nabla \cdot (b y_{\boldsymbol{\sigma}}) &= \sum_{i=1}^{N_{\mathrm{a}}} u_i \mathbf{1}_{O_i} && (t,\xi) \in (0,T] \times D, \\
\frac{\partial y_{\boldsymbol{\sigma}}}{\partial \mathrm{n}} &= 0 && (t,\xi) \in [0,T] \times \partial D, \\
y_{\boldsymbol{\sigma}} &= y_{\circ} && (t,\xi) \in \{t=0\} \times D,
\end{aligned}
$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_s) \in \left[-\frac{1}{2}, \frac{1}{2}\right]^s$ enters the diffusion as

$$
a_{\boldsymbol{\sigma}}(\xi) = \bar{a}(\xi) + \sum_{j=1}^{s} \boldsymbol{\sigma}_j \psi_j(\xi),
$$

with $\bar{a} \in L^\infty(D)$ and $\psi_j \in L^\infty(D)$ $\forall 1 \leq j \leq s$ such that, there exist $a_{\min}$ and $a_{\max}$ satisfying

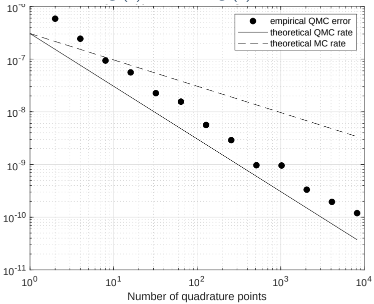
$$
0 < a_{\min} \leq a_{\boldsymbol{\sigma}}(\xi) \leq a_{\max} < \infty \quad \text{for all } \xi \in D \text{ and } \boldsymbol{\sigma} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^s.
$$

Numerical experiments setup

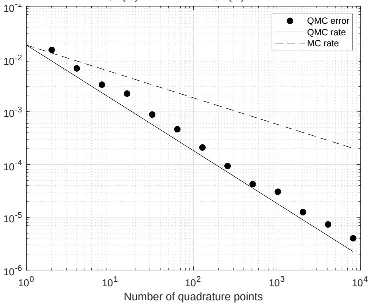
- $D = (0, 1)$ and $T = 5$
- $N_a = 3$ actuators as $O_1 = [0.1, 0.3]$, $O_2 = [0.4, 0.6]$, and $O_3 = [0.7, 0.9]$
- constant reaction coefficient $c = -1$, no convection $b = 0$
- initial condition $y_\circ(s) = \sin(2\pi s) - 1$
- the target g solves $\dot{g} = 0.1\Delta g$ with the same data
- stochastic dimension $s = 100$
- $\bar{a} = 1.15$, with basis functions $\psi_{2j}(s) = \eta(2j)^{-\vartheta}\sin(j\pi s)$ and $\psi_{2j-1}(s) = \eta(2j-1)^{-\vartheta}\cos(j\pi s)$ with $\vartheta \in \{2, 3\}$ and $\eta \in \{0.1, 1\}$.
- QMC points generated using `https://people.cs.kuleuven.be/~dirk.nuyens/qmc4pde/`

Randomly shifted rank 1 lattice rule

$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^* \Pi_{\boldsymbol{\sigma}}(t)$



$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^* h_{\boldsymbol{\sigma}}(t)$

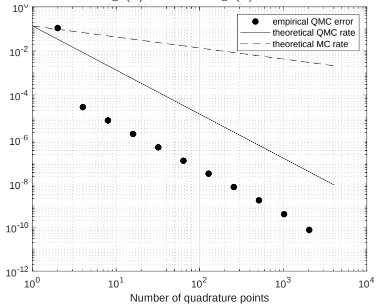


$$RMSE \approx \sqrt{\frac{1}{R(R-1)} \sum_{r=1}^{R} \left\| \left(\overline{Q} - Q^{(r)} \right) (\mathscr{K}) \right\|_Z^2},$$

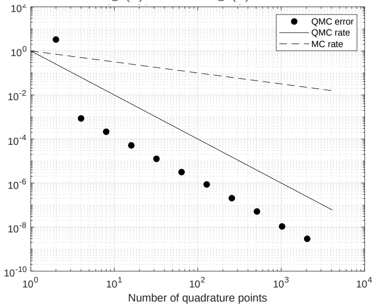
where $Q^{(r)}(\mathscr{K}) := \frac{1}{N} \sum_{k=1}^{N} \mathscr{K}(\boldsymbol{\sigma}_{\Delta^{(r)}}^{(k)})$ and $\overline{Q}(\mathscr{K}) := \frac{1}{R} \sum_{r=1}^{R} Q^{(r)}(\mathscr{K})$.

Interlaced polynomial lattice rule with $\alpha = 2$

$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^* \Pi_{\boldsymbol{\sigma}}(t)$

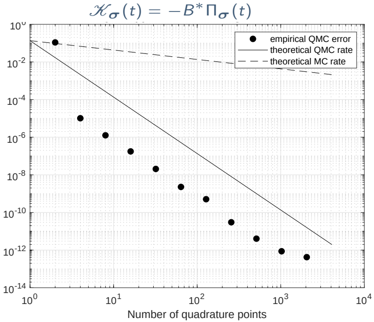


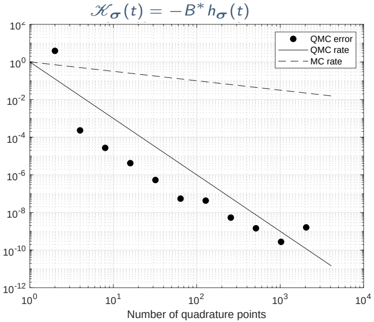
$\mathscr{K}_{\boldsymbol{\sigma}}(t) = -B^* h_{\boldsymbol{\sigma}}(t)$



$$\left\| Q_{\mathrm{ref}}(\mathscr{K}_{\boldsymbol{\sigma},s}(t)) - \frac{1}{N} \sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}^{(k)},s}(t) \right\|_Z, \quad \text{where we used } 2^{12} \text{ points for } Q_{\mathrm{ref}}.$$

Interlaced polynomial lattice rule with $\alpha = 3$





$$\left\| Q_{\mathrm{ref}}(\mathscr{K}_{\boldsymbol{\sigma},s}(t)) - \frac{1}{N}\sum_{k=0}^{N-1} \mathscr{K}_{\boldsymbol{\sigma}^{(k)},s}(t) \right\|_Z, \quad \text{where we used } 2^{12} \text{ points for } Q_{\mathrm{ref}}.$$

Error propagation

Theorem (Feedback approximation error propagation)

Let $\hat{K} := \sum_{k=1}^{N} \alpha_k K_{\boldsymbol{\sigma}_k,s}(t,\cdot)$ *be an approximation of the feedback* $K = \int_{\mathfrak{S}} K_{\boldsymbol{\sigma}}(t,\cdot)\,\mathrm{d}\boldsymbol{\sigma}$*. Then, there holds for all* $t \in [0,T]$

$$\|y_{\boldsymbol{\sigma}}(t) - \hat{y}_{\boldsymbol{\sigma}}(t)\|_H \leq C_y \max_{t\in[0,T]} \left(\|B^*\delta\Pi(T-t)\|_{\mathcal{L}(H,U)} + \|B^*\delta h(t)\|_U\right),$$

$$\|u_{\boldsymbol{\sigma}}(t) - \hat{u}_{\boldsymbol{\sigma}}(t)\|_U \leq C_u \max_{t\in[0,T]} \left(\|B^*\delta\Pi(T-t)\|_{\mathcal{L}(H,U)} + \|B^*\delta h(t)\|_U\right),$$

where the constants C_u *and* C_y *are independent of* $\boldsymbol{\sigma} \in \mathfrak{S}$*.*

We use the notation: $B^*\Pi := \int_{\mathfrak{S}} B^*\Pi_{\boldsymbol{\sigma}}\,\mathrm{d}\boldsymbol{\sigma}$, $B^*\hat{\Pi} := \sum_{k=1}^{N} \alpha_k B^*\Pi_{\boldsymbol{\sigma}_k,s}$, $B^*\delta\Pi := B^*\Pi - B^*\hat{\Pi}$, $B^*h := \int_{\mathfrak{S}} B^*h_{\boldsymbol{\sigma}}\,\mathrm{d}\boldsymbol{\sigma}$, $B^*\hat{h} := \sum_{k=1}^{N} \alpha_k B^*h_{\boldsymbol{\sigma}_k,s}$, and $B^*\delta h = B^*h - B^*\hat{h}$.

Suboptimality of the feedback

Suppose there is a 'true' parameter $\bar{\boldsymbol{\sigma}} \in \mathfrak{S}$ with optimal feedback $K_{\bar{\boldsymbol{\sigma}}}(t,\cdot) = -B^*\Pi_{\bar{\boldsymbol{\sigma}}}(T-t)(\cdot) - B^* h_{\bar{\boldsymbol{\sigma}}}(t)$.

Theorem

The trajectories are close provided that K is close to $K_{\bar{\boldsymbol{\sigma}}}$

$$\|y_{\bar{\boldsymbol{\sigma}}}(t) - y_{\boldsymbol{\sigma}}(t)\|_H \leq \bar{C}_y \max_{t\in[0,T]} \left(\|B^*\delta\Pi_{\bar{\boldsymbol{\sigma}}}(T-t)\|_{\mathcal{L}(H,U)} + \|B^*\delta h_{\bar{\boldsymbol{\sigma}}}(t)\|_U\right),$$

$$\|u_{\bar{\boldsymbol{\sigma}}}(t) - u_{\boldsymbol{\sigma}}(t)\|_U \leq \bar{C}_u \max_{t\in[0,T]} \left(\|B^*\delta\Pi_{\bar{\boldsymbol{\sigma}}}(T-t)\|_{\mathcal{L}(H,U)} + \|B^*\delta h_{\bar{\boldsymbol{\sigma}}}(t)\|_U\right),$$

for all $t \in [0,T]$ where the constants $\bar{C}_u$ and $\bar{C}_y$ are independent of $\boldsymbol{\sigma} \in \mathfrak{S}$.

We use the notation: $B^*\delta\Pi_{\bar{\boldsymbol{\sigma}}} := B^*\Pi_{\bar{\boldsymbol{\sigma}}} - B^*\Pi$, and $B^*\delta h_{\bar{\boldsymbol{\sigma}}} = B^* h_{\bar{\boldsymbol{\sigma}}} - B^* h$

Conclusions

- Construction of feedback is independent of initial condition
- Feedback is independent of a particular realization of the parameter, thus it can be computed *a-priori*.
- Analytic regularity of feedback operator
- General QMC framework allows integration of operators

Thank you!