Multiindex Monte Carlo for semilinear SPDE

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Problem description

Seek mild solutions of H-valued SDE

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t)\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t), \quad t \in (0, T]$$
$$X(0) = X_0 \in H$$

Goal: Approximate $\mathbb{E}[\Psi(X(T))]$ for smooth QoI $\Psi : H \to U$.

Contribution: A multiindex Monte Carlo method

$$\mu_{MI} := \sum_{\boldsymbol{\ell} \in \mathcal{I} \subset \mathbb{N}_0^2} \sum_{i_{\boldsymbol{\ell}}=1}^{m_{\boldsymbol{\ell}}} \frac{\Psi(X_{N_{\ell_2}}^{M_{\ell_1}, i_{\boldsymbol{\ell}}}) - \Psi(X_{N_{\ell_2}}^{M_{\ell_1-1}, i_{\boldsymbol{\ell}}}) - \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1}, i_{\boldsymbol{\ell}}}) + \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1-1}, i_{\boldsymbol{\ell}}})}{m_{\boldsymbol{\ell}}}$$

that achieves

$$\mathbb{E}[\|\mu_{MI} - \mathbb{E}[\Psi(X(T))]\|_U^2] \lesssim \epsilon^2$$

at a low computational cost.

Overview

The SPDE and mild solutions

Monte Carlo methods for approximating $\mathbb{E}[\Psi(X(T))]$

Exponential integrator method

Numerical experiments and conclusion

The SPDE and mild solutions

The SPDE

We consider

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t))\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t)$$

on Hilbert space $H = (H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ and where: $A : \mathrm{Dom}(A) \subset H \to H$ is a

densely defined, positive definite linear operator with orthonormal eigenbasis

$$\big((e_j, \lambda_j)\big)_{j=1}^{\infty}, \qquad \text{where } \lambda_j \eqsim j^{\nu} \quad \text{for some } \nu > 0$$

Fractional operators $\qquad A^r v := \sum_{j=1}^{\infty} \lambda_j^r \langle e_j, v\rangle e_j \quad \text{for} \quad r \in \mathbb{R}$

Extension of norm: $\|\cdot\|_{\dot{H}_r} := \|A^{r/2}\cdot\|$ and associated Hilbert space

$$\dot{H}^r = \begin{cases} \mathrm{Dom}(A^{r/2}) & r \geq 0 \\ \overline{H}^{\|\cdot\|_{\dot{H}^r}} & r < 0, \end{cases}$$

Canonical example: Stochastic heat equation

We let $H = L^2(\mathcal{D})$ for a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, convex or with $\mathcal{C}^2$ boundary $\partial\mathcal{D}$.

Let $A = -\Delta$ with homogeneous zero Dirichlet boundary conditions. Then $\lambda_j \eqsim j^{2/d}$.

For $d = 1$ with $\mathcal{D} = (0, 1)$ eigenpairs then are

$$e_k(x) = \sqrt{2}\sin(k\pi x), \qquad \text{and} \quad \lambda_k = \pi^2 k^2.$$

Drift coefficient: Natural example of $F : H \to H$ is a composition (Nemytskij) mapping

$$(F(u))(x) = f(u(x))$$

where $f : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth with bounded derivatives.

Q-Wiener process

W is a $Q-$Wiener process, meaning it has independent increments

$$W(t + \Delta t) - W(t) \sim N(0, \Delta t Q).$$

We assume $Q \in \mathcal{L}_1(H)$ with eigenpairs (q_k, e_k) (same eigenbasis as operator A!).

Representation: $$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j} B_j(t) e_j$$

with $B_j(t)$ independent, scalar-valued Brownian motions, defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$.

The noise operator G is specified later.

Q-Wiener process

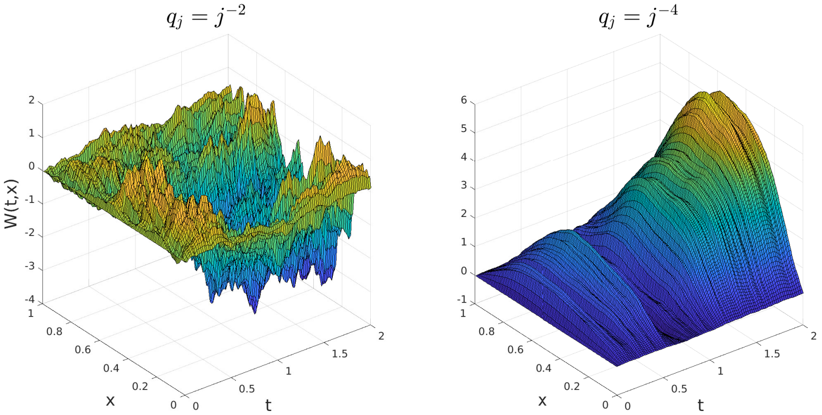

Figure 1: Regularity of W depends on decay rate of (q_j)

Mild solution[1]

If X_0 is $\mathcal{F}_0-$measurable and $X_0 \in L^p(\Omega, H)$, then there exists a unique mild solution in $C([0,T], L^p(\Omega, H))$ to the SPDE

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t)\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t), \quad t \in (0,T]$$
$$X(0) = X_0.$$

Definition: a mild solution is an H-valued predictable process $\{X(t)\}_{t\in[0,T]}$ satisfying

$$X(t) = e^{-At}X_0 + \int_0^t e^{-A(t-s)}F(X(s))\,\mathrm{d}s + \int_0^t e^{-A(t-s)}\big(I + GX(s)\big)\,\mathrm{d}W(s)$$

for each $t \in [0,T]$.

[1] Da Prato, G. & Zabczyk, J. *Stochastic equations in infinite dimensions.* Second, xviii+493 (Cambridge University Press, Cambridge, 2014).

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Multilevel Monte Carlo (MLMC)[2]

Let X_N^M be a *general* numerical method with respect to integers M (e.g. timesteps) and N (e.g. eigenfunctions).

For increasing $(M_\ell)_{\ell=0}^\infty, (N_\ell)_{\ell=0}^\infty \subset \mathbb{N}$, consider the telescoping sum

$$\mathbb{E}[\Psi(X_{N_L}^{M_L}(T))] = \sum_{\ell=1}^{L} \mathbb{E}[\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))] + \mathbb{E}[\Psi(X_{N_0}^{M_0}(T))].$$

This motivates MLMC estimator based on sample averages E_{m_ℓ},

$$\mu_{ML} := \sum_{\ell=1}^{L} E_{m_\ell}[\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))] + E_{m_0}[\Psi(X_{N_0}^{M_0}(T))].$$

We assume that (here and below we ignore logarithmic terms!):

$$\mathrm{Cost}(\Psi(X_N^M)) \eqsim MN.$$

[2] Giles, M. B. Multilevel monte carlo path simulation. *Operations research* **56,** 607–617 (2008).

Multilevel Monte Carlo (MLMC)

Central point: given that

$$\|X(T) - X_N^M(T)\|^2_{L^2(\Omega,H)} \lesssim M^{-1} + N^{-\beta}$$

for some $\beta \geq 1$, then with $M_\ell = 2^\ell$ and $N_\ell \eqsim 2^{\ell/\beta}$

$$\|\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))\|^2_{L^2(\Omega,U)} \lesssim 2^{-\ell}$$

meaning very few samples m_ℓ needed when $\ell \gg 1$.

Performance[3]: for any $\epsilon > 0$, there exists $(m_\ell) \subset \mathbb{N}$ s.t.

$$\mathbb{E}[\|\mu_{ML} - \mathbb{E}[\Psi(X(T))]\|^2_U] \lesssim \epsilon^2$$

with $\mathrm{Cost}(\mu_{ML}) \eqsim \epsilon^{-2-\max\left(\frac{1}{\beta\alpha_1},\frac{1}{\alpha_2}\right)}$ for *weak convergence rates* α_1, α_2.

[3]Chada, N. K., Hoel, H., Jasra, A. & Zouraris, G. E. Improved efficiency of multilevel Monte Carlo for stochastic PDE through strong pairwise coupling. *J. Sci. Comput.* **93,** Paper No. 62, 29 (2022).

Multiindex Monte Carlo (MIMC)[4]

Big improvement over classic Monte Carlo. Can we do better with the *multiindex* Monte Carlo method?

Let $M_k \eqsim N_k \eqsim 2^k$ be sequences of integers and consider telescoping double-sum

$$\mathbb{E}[\Psi(X_{N_L}^{M_L}(T))] = \sum_{\ell_1=0}^{L}\sum_{\ell_2=0}^{L}\mathbb{E}[\ \underbrace{\Psi(X_{N_{\ell_2}}^{M_{\ell_1}}) - \Psi(X_{N_{\ell_2}}^{M_{\ell_1-1}}) - \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1}}) + \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1-1}})}_{=:\Delta_{\boldsymbol{\ell}}\Psi(X)}\ ]$$

with convention $\Psi(X_{N_{\ell_2-1}}^{M_{\ell_1-1}}(T)) := 0$ whenever $\min(\ell_1, \ell_2) = 0$.

This motivates the MIMC estimator

$$\mu_{MI} := \sum_{\boldsymbol{\ell}\in\mathcal{I}} E_{m_{\boldsymbol{\ell}}}[\Delta_{\boldsymbol{\ell}}\Psi(X)]$$

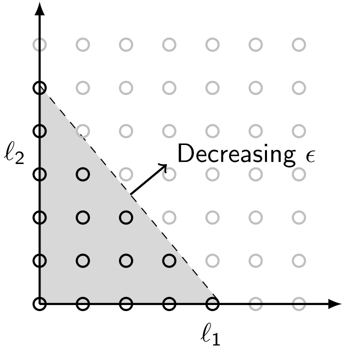
where $\mathcal{I} = \{\boldsymbol{\ell} = (\ell_1, \ell_2) \in \mathbb{N}_0^2 \mid \max(\ell_1, \ell_2) \leq L\}$ and $(m_{\boldsymbol{\ell}}) \subset \mathbb{N}$.

[4] Haji-Ali, A.-L., Nobile, F. & Tempone, R. Multi-index Monte Carlo: when sparsity meets sampling. *Numerische Mathematik* **132,** 767–806 (2016).

The index set

Triangular index sets are more efficent than rectangular ones, so for some suitable weights (w_1, w_2) we actually use:

$$\mathcal{I} = \{\ell \in \mathbb{N}_0^2 | w_1\ell_1 + w_2\ell_2 \lesssim L(\epsilon)\}$$



MIMC for SPDE for general numerical method

Theorem

Suppose that $\Psi(X_N^M(T)) \to \Psi(X(T))$ *as* $M, N \to \infty$. *Assume also that we have* **a multiplicative** *bound on the* **second order difference**

$$\|\Psi(X_{\bar{N}}^{\bar{M}}(T)) - \Psi(X_N^{\bar{M}}(T)) - \Psi(X_{\bar{N}}^{M}(T)) + \Psi(X_N^M(T))\|^2_{L^2(\Omega,U)} \lesssim M^{-1}N^{-1}$$

for $\bar{M} \geq M, \bar{N} \geq N$. *Then there exist MIMC parameters* $\mathcal{I}$ *and* $(m_\ell) \subset \mathbb{N}$ *s.t.*

$$\|\mu_{MI} - \mathbb{E}[\Psi(X(T))]\|^2_{L^2(\Omega,U)} \lesssim \epsilon^2$$

at "optimal" Monte Carlo cost $\mathrm{Cost}(\mu_{MI}) \eqsim \epsilon^{-2}$.

NB! Suitable numerical method may be difficult to find[5].

[5] Reisinger, C. & Wang, Z. Analysis of multi-index Monte Carlo estimators for a Zakai SPDE. English. *J. Comput. Math.* **36,** 202–236 (2018).

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Exponential integrator method[6]

Discretizations: $\Delta t = T/M$ and let $P_N : H \to \mathrm{Span}(e_1, \ldots, e_N) =: H_N$ be orthogonal projection.

Spectral numerical method $X_N^M : [0, T] \to H_N$ given by

$$X_N^M(t_{j+1}) = e^{-A\Delta t} X_N^M(t_j) + \int_{t_j}^{t_{j+1}} e^{-A(t_{j+1}-s)} P_N F(X_N^M(t_j))\,\mathrm{d}s$$
$$+ \int_{t_j}^{t_{j+1}} e^{-A(t_{j+1}-s)} P_N(I + G X_N^M(t_j)) dW(s)$$
$$X_N^M(0) = P_N X_0$$

Note: Q and A share an eigenbasis and when F is nonlinear, iterations can often be solved using FFT.

[6] Jentzen, A. & Kloeden, P. E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. English. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* **465,** 649–667 (2009).

Motivation

For exponential integrator method, one obtains that

$$\begin{aligned}
&X_{\bar{N}}^{\bar{M}}(T) - X_{N}^{\bar{M}}(T) - X_{\bar{N}}^{M}(T) + X_{N}^{M}(T) \\
&\quad = e^{-AT}(P_{\bar{N}} - P_N - P_{\bar{N}} + P_N)X_0 \\
&\qquad + \int_0^T e^{-A(T-s)}[\text{second order difference for } F](s)\,\mathrm{d}s \\
&\qquad + \int_0^T e^{-A(T-s)}[\text{second order difference for } G](s)\,\mathrm{d}W(s) \\
&\qquad + \int_0^T e^{-A(T-s)}(P_{\bar{N}} - P_N - P_{\bar{N}} + P_N)\,\mathrm{d}W(s).
\end{aligned}$$

Neither the initial term nor the additive stochastic term directly contribute to either the spatial or the temporal part of the error!

Assumptions on F

Let $F \in \mathcal{G}^2(H, H)$ and

(i) $\|F'(u)v\| \leq C\|v\|, \quad u, v \in H,$

(ii) $\|F'(u)v\| \leq C(1 + \|u\|_{\dot{H}^1})\|v\|_{\dot{H}^{-1}}, \quad u \in \dot{H}^1, v \in H$

(iii) $\|F''(u)(v_1, v_2)\|_H \leq C\|v_1\|\|v_2\|, \quad u, v_1, v_2 \in H.$

Assumptions on X_0, Q, G:

(iv) $X_0 \in L^p(\Omega, \dot{H}^1)$ $\mathcal{F}_0$-measurable with $p \geq 10$

(v) $Q \in \mathcal{L}_1(H)$, i.e., $\sum_j q_j < \infty$

(vi) G is an operator on the spectrum of A:

$$(Gu)v := \sum_{j=1}^{\infty} \alpha_j \langle u, e_{j+m} \rangle \langle v, e_j \rangle e_j, \quad u \in H, v \in Q^{1/2}(H)$$

for a shift $m \in \mathbb{N}_0$ and sequence $(\alpha_j)_{j=1}^{\infty} \subset \mathbb{R}$ fulfilling $\lim_{j \to \infty} q_j^{1/2} |\alpha_j| = 0$.

Regularity and error estimates

Recall that $\lambda_k \eqsim k^\nu$ and that $M_k \eqsim N_k \eqsim 2^k$. Given the assumptions above:

Theorem

Recall that $\lambda_k \eqsim k^\nu$ and that $M_k \eqsim N_k \eqsim 2^k$. Given the assumptions above:

(i) $\sup_j \|X_{N_{\ell_2}}^{M_{\ell_1}}(t_j)\|^2_{L^p(\Omega,\dot{H}^1)} < \infty$

(ii) $\|X_{N_{\ell_2+1}}^{M_{\ell_1}}(T) - X_{N_{\ell_2}}^{M_{\ell_1}}(T))\|^2_{L^p(\Omega,H)} \lesssim N_{\ell_2}^{-\nu}$,

(iii) $\|X_{N_{\ell_2}}^{M_{\ell_1+1}}(T) - X_{N_{\ell_2}}^{M_{\ell_1}}(T)\|^2_{L^p(\Omega,H)} \lesssim M_{\ell_1}^{-1}$ *and*

(iv)

$$\|\Delta_{\boldsymbol{\ell}} X\|^2_{L^2(\Omega,H)} \lesssim \min(M_{\ell_1}^{-1} N_{\ell_2}^{-\nu},\, N_{\ell_2}^{-2\nu}) + 1_{\ell_1=0} N_{\ell_2}^{-\nu}$$

(v) *And for sufficiently smooth QoI $\Psi \in \mathcal{G}^2(H,U)$,*

$$\|\Delta_{\boldsymbol{\ell}} \Psi(X)\|^2_{L^2(\Omega,U)} \lesssim M_{\ell_1}^{-1} N_{\ell_2}^{-\nu}.$$

MIMC cost versus error

Theorem

Let $\alpha_1, \alpha_2 > 0$ be weak rates of double difference

$$\|\mathbb{E}[\Delta_{\boldsymbol{\ell}}\Psi(X)]\|_U \lesssim M_{\ell_1}^{-\alpha_1} N_{\ell_2}^{-\alpha_2}$$

Then $\alpha_1 \geq 1/2$ and $\alpha_2 \geq \nu/2$, and there exist MIMC parameters $\mathcal{I}$ and $(m_{\boldsymbol{\ell}}) \subset \mathbb{N}$ s.t.

$$\|\mu_{MI} - \mathbb{E}[\Psi(X(T))]\|_{L^2(\Omega,U)}^2 \lesssim \epsilon^2$$

at

$$\mathrm{Cost}(\mu_{MI}) \lesssim \begin{cases} \epsilon^{-2} & \textit{if} \quad \nu \geq 1 \\ \epsilon^{-2-(1-\nu)/\alpha_2} & \textit{if} \quad \nu \in (0,1) \end{cases}$$

Potential efficiency gains:

$$\frac{\mathrm{Cost}(\mu_{MI})}{\mathrm{Cost}(\mu_{ML})} = \begin{cases} \mathcal{O}(\epsilon^{2/\nu}) & \text{if} \quad \nu \geq 1 \\ \mathcal{O}(\epsilon^2) & \text{if} \quad \nu \in (0,1) \end{cases}$$

Overview

Verification of multiplicative convergence rate

Test for QoI $\Psi(x) = x$ and seek to verify multiplicative convergence property

$$\|\Delta_{\boldsymbol{\ell}} X\|_{L^2(\Omega,H)} \approx \sqrt{E_{M=10^4}[\|\Delta_{\boldsymbol{\ell}} X\|^2]} =: e_F(\ell_1, \ell_2)$$

with $M_\ell \eqsim N_\ell \eqsim 2^\ell$.

If our theoretical rates are sharp, then

$$\|\Delta_{\boldsymbol{\ell}} X\|_{L^2(\Omega,H)} \eqsim \min(2^{-\ell_1/2-(\nu/2)\ell_2}, 2^{-\nu\ell_2})$$

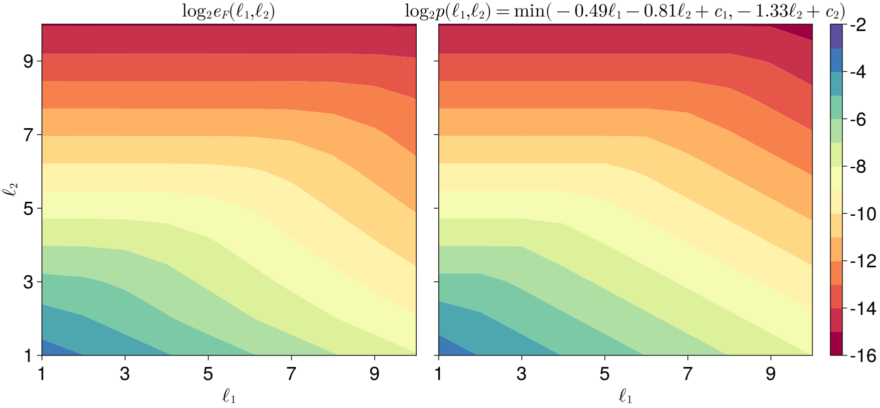
Numerical verification: Find $\beta_1, \beta_2, \beta_3 > 0$ by a least square fit, such that

$$p(\ell_1, \ell_2) := C \min(2^{-\beta_1\ell_1/2-\beta_2\ell_2/2}, 2^{-\beta_3\ell_1})$$

dominates $e_F(\ell_1, \ell_2)$ and verify that $\beta_1 \approx 1$, $\beta_2 \approx \nu$ and $\beta_3 \approx \nu$.

Numerical test I – linear $F(X)$

SPDE $dX = (0.2(-\Delta)^{2/3} + X)dt + (I + GX)dW$ with $q_j \approx j^{-1.01}$ on $\mathcal{D} = (0,1)$, so $\nu = 4/3$.

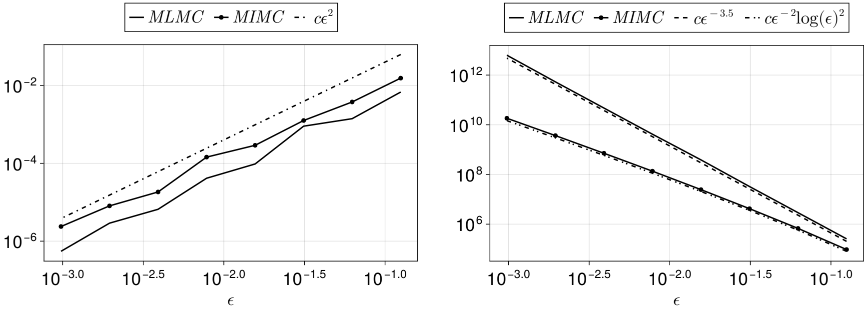


Left: $e_F(\ell_1,\ell_2)$. **Right:** $p(\ell_1,\ell_2)$ with $\beta_1 = 0.98$, $\beta_2 = 1.62$ and $\beta_3 = 1.33$.

Expected rates: $\beta_1 = 1$ and $\beta_2 = \beta_3 = 4/3$.

Numerical test II

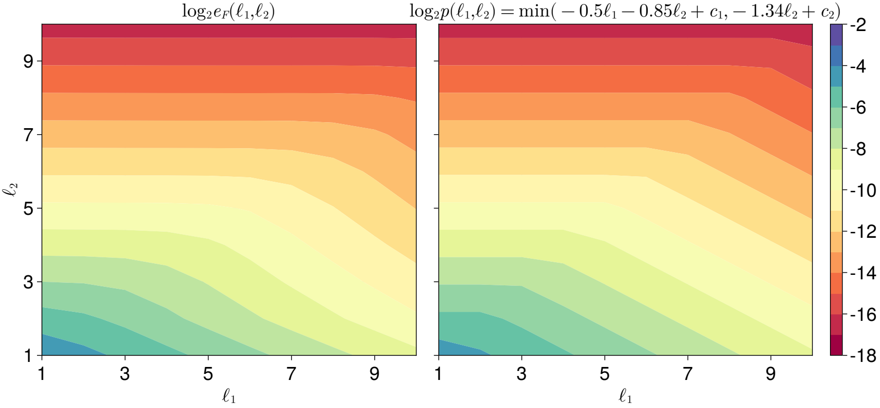
Performance comparison MLMC vs MIMC, linear case.



Left: Error $\|\mu_{MI}(\epsilon) - \mathbb{E}[X(1)]\|^2$. **Right:** $\mathrm{Cost}(\mu_{MI}(\epsilon))$

Numerical test III – nonlinear $F(X)$

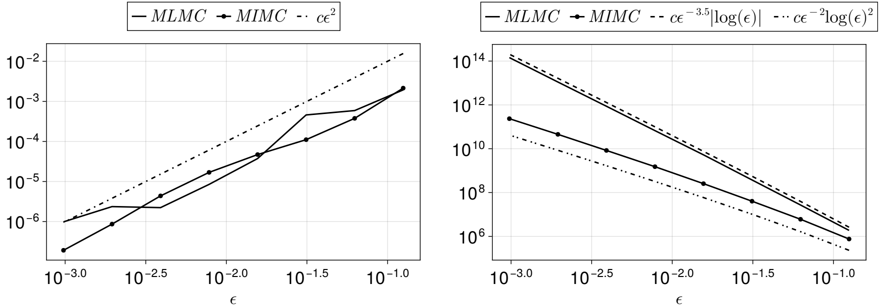
SPDE $dX = (-\Delta)^{2/3} + \sin(X))dt + (I + G(X))dW$ with $q_j \eqsim j^{-1.01}$ on $\mathcal{D} = (0,1)$, so $\nu = 4/3$.



Left: $e_F(\ell_1,\ell_2)$. **Right:** $p(\ell_1,\ell_2)$ with $\beta_1 = 1$, $\beta_2 = 1.7$ and $\beta_3 = 1.34$.

Expected rates: $\beta_1 = 1$ and $\beta_2 = \beta_3 = 4/3$.

Performance comparison MLMC vs MIMC, nonlinear case.



Left: Error $\|\mu_{MI}(\epsilon) - \mathbb{E}[X(1)]\|^2$. **Right:** $\mathrm{Cost}(\mu_{MI}(\epsilon))$

Summary and future work

- Developed efficent multiindex Monte Carlo method for approximations of semilinear SPDE
- Future work: Study if recent splitting methods[7] in setting with $G = 0$ can be combined with MIMC to handle less regular f, e.g., Allen–Cahn.
- Restriction: Operators Q and A have to share eigenbasis (e_k) on which G acts
- Open problem: Analysis does not cover error contribution from FFT to evalutate $F(X_N^M)$.

[7] Djurdjevac, A., Gerencsér, M. & Kremp, H. Higher order approximation of nonlinear SPDEs with additive space-time white noise. *arXiv preprint arXiv:2406.03058.* arXiv: `2406.03058 [math.PR]`. `https://doi.org/10.48550/arXiv.2406.03058` (2024).

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Thank you for listening!

Assumptions on F

Original assumptions[8]: $F\colon H \to H$ is twice Fréchet differentiable and

$$\|F'(u)v\|_{\dot{H}^{-r}} \leq C\|v\|_{\dot{H}^{-r}}, \quad r \in \{0,1,2\}, u, v \in H$$
$$\|F''(u)(v,w)\|_{\dot{H}^{-2}} \leq C\|v\|_{\dot{H}^{-1}}\|w\|_{\dot{H}^{-1}}, u, v \in H$$

for C ind. of u. If F is a composition operator in $L^2(\mathcal{D})$, *this implies linearity*.

Instead: Let $F \in \mathcal{G}^1(H,H) \cap \mathcal{G}^2(H, \dot{H}^{-\eta})$ for some $\eta \in [0,2)$. Moreover, for a *regularity parameter* $\kappa \in (0,2)$:

(i) $\|F'(u)v\| \leq C\|v\|, \quad u, v \in H,$

(ii) $\|F(u) - F(v)\|_{\dot{H}^{\kappa-\delta}} \leq C(1 + \|u\|^2_{\dot{H}^\kappa} + \|v\|^2_{\dot{H}^\kappa}), \quad u, v \in \dot{H}^\kappa,$

(iii) $\|F'(u)v\|_{\dot{H}^{-\eta}} \leq C(1 + \|u\|^{\lceil r \rceil}_{\dot{H}^\kappa})\|v\|_{\dot{H}^{-r}}, r \in \{1, \kappa - \delta\}, \quad u \in \dot{H}^\kappa, v \in H$

(iv) $\|F''(u)(v_1, v_2)\|_{\dot{H}^{-\eta}} \leq C\|v_1\|\|v_2\|, \quad u, v_1, v_2 \in H.$

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for C ind. of u. If F is a composition operator in $L^2(\mathcal{D})$, *this implies linearity*.

Instead (for $\eta = 0$ and $\kappa = 1$): Let $F \in \mathcal{G}^1(H,H) \cap \mathcal{G}^2(H,H)$:

(i) $\|F'(u)v\| \le C\|v\|, \quad u, v \in H,$

(ii) $\|F(u) - F(v)\|_H \le C(1 + \|u\|_{\dot{H}^1}^2 + \|v\|_{\dot{H}^1}^2), \quad u, v \in \dot{H}^1,$

(iii) $\|F'(u)v\|_H \le C(1 + \|u\|_{\dot{H}^1})\|v\|_{\dot{H}^{-1}}, \quad u \in \dot{H}^1, v \in H$

(iv) $\|F''(u)(v_1, v_2)\|_H \le C\|v_1\|\|v_2\|, \quad u, v_1, v_2 \in H.$

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Assumptions on X_0, Q, G

For some $p \geq 10$:

(v) $X_0 \in L^p(\Omega, \dot{H}^\kappa)$ $\mathcal{F}_0$-measurable with $p \geq 10$

(vi) $Q^{1/2}(H) \overset{\mathcal{L}_2}{\hookrightarrow} \dot{H}^{\kappa-1}$, i.e., $\sum_j \lambda_j^{\kappa-1} q_j < \infty$

(vii) G is an operator on the spectrum of A:

$$(Gu)v := \sum_{j=1}^{\infty} \alpha_j \langle u, e_{j+m} \rangle \langle v, e_j \rangle e_j, \quad u \in H, v \in Q^{1/2}(H)$$

for a shift $m \in \mathbb{N}_0$ and a sequence $(\alpha_j)_{j=1}^\infty \subset \mathbb{R}$ fulfilling $|\alpha_j| \lesssim (\lambda_j^{\kappa-1} q_j)^{-1/2}$

Other notions of solutions

An H-valued predictable process $\{X(t)\}_{t\in[0,T]}$ is called:

- a **strong solution** of the SPDE if for all $t \in (0, T]$,

$$X(t) = X_0 + \int_0^t F(X(s) - AX(s)\,\mathrm{d}s + \int_0^t (I + GX(s))\,\mathrm{d}W(s).$$

  Problem: Need that $X \in \mathrm{Dom}(A)$, and often it is not that smooth.

- a **weak solution** of the SPDE if for all $t \in (0, T]$ and $v \in \mathrm{Dom}(A)$

$$\begin{aligned}\langle X(t), v\rangle =& \langle X_0, v\rangle + \int_0^t \langle F(X(s), v\rangle - \langle X(s), Av\rangle\,\mathrm{d}s \\ &+ \int_0^t \langle (I + GX(s))\,\mathrm{d}W(s), v\rangle.\end{aligned}$$

Relationship[9]**:** Strong solutions are weak solutions and weak solutions are typically mild solutions.

[9] Liu, W. & Röckner, M. *Stochastic partial differential equations: an introduction.* (Springer, 2015).

Motivation

Regularity: The semigroup e^{-At} is smoothing:

$$\|e^{-At}\|_{\mathcal{L}(H)} = \|e^{-At}e_1\| = e^{-\lambda_1 t} < 1$$

So that

$$\|e^{-At}X_0\|_{\dot{H}^1} = \|A^{1/2}e^{-At}X_0\| \leq \|e^{-At}A^{1/2}X_0\| \leq \|A^{1/2}X_0\|.$$

And $\|A^{1/2}e^{-At}\|_{\mathcal{L}(H)} \leq Ct^{-1/2}$ used to bound $\dot{H}^1$-norm of other terms, e.g.,

$$\begin{aligned}\int_0^t \|e^{A(t-s)}P_N F(X_N^M(s))\|_{L^p(\Omega,\dot{H}^1)}\,\mathrm{d}s \\ &= \int_0^t \|A^{1/2}e^{A(t-s)}P_N F(X_N^M(s))\|_{L^p(\Omega,H)}\,\mathrm{d}s \\ &\leq c\int_0^t (t-s)^{1/2}\|P_N F(X_N^M(s))\|_{L^p(\Omega,H)}\,\mathrm{d}s\end{aligned}$$

Motivation

Numerical error: bound by

$$\|X(T) - X_N^M(T)\| \leq \underbrace{\|X(T) - P_N X(T)\|}_{\text{spatial error}} + \underbrace{\|P_N X(T) - X_N^M(T)\|}_{\text{time error}}.$$

Spatial error:

$$\|(I - P_N)X(T)\| = \|A^{-1/2}(I - P_N)A^{1/2}X(T)\| \leq \|A^{-1/2}(I - P_N)\|_{\mathcal{L}(H)}$$

and

$$\|A^{-1/2}(I - P_N)\|_{\mathcal{L}(H)} = \|A^{-1/2}(I - P_N)e_{N+1}\| = \lambda_{N+1}^{-1/2}$$

Time error:

$$\|P_N X(T) - X_N^M(T)\|_{L^p(\Omega,H)} = \mathcal{O}(\sqrt{\Delta t})$$

is same rate as Euler–Maruyama has for N-dimensional SDE.

Computation: The exponential integrator approximation X_N^M of (8) is now given by $X_N^M(0) := P_N X_0$ and for $j = 0, \dots, M-1$ by

$$X_N^M(t_{j+1}) := S(\Delta t) X_N^M(t_j) + \int_{t_j}^{t_{j+1}} S(t_{j+1} - s) P_N F(X_N^M(t_j)) \, \mathrm{d}s + \int_{t_j}^{t_{j+1}} S(t_{j+1} - s) \big(P_N + G X_N^M(t_j) \big) \, \mathrm{d}W(s).$$

Note that

$$\Big\langle \int_{t_j}^{t_{j+1}} S(t_{j+1} - s) \big(P_N + G X_N^M(t_j) \big) \, \mathrm{d}W(s), e_k \Big\rangle = q_k^{\frac{1}{2}} (1 + \alpha_k \langle X_N^M(t_j), e_{k+m} \rangle) \int_{t_j}^{t_{j+1}} e^{-\lambda_k (t_{j+1} - s)} \, \mathrm{d}B_k(s)$$

if $k \leq N$ and 0 otherwise, meaning the stochastic term can be sampled exactly.

Proof ideas

Repeated use of the mean value theorem, e.g.,

$$
\begin{aligned}
&\big(F(X_{\bar N}^{\bar M}(t_j) + X_N^M(t_j) - X_{\bar N}^M(t_j)) - F(X_{\bar N}^{\bar M}(t_j))\big) - \big(F(X_N^M(t_j)) - F(X_{\bar N}^M(t_j))\big)\\
&= \int_0^1 \int_0^1 F''(X_{\bar N}^M(t_j) + \lambda(\cdots) + \tilde\lambda(\cdots))\big(X_{\bar N}^{\bar M}(t_j) - X_{\bar N}^M(t_j)\big)\big(X_N^M(t_j) - X_{\bar N}^M(t_j)\big)\,\mathrm{d}\lambda\,\mathrm{d}\tilde\lambda,
\end{aligned}
$$

BDG inequality, smoothing of the semigroup, negative norm bounds on single errors and standard bounds on $I - P_N$, $I - S(\Delta t)$.

Negative norm bounds on F,

$$\|F'(u)v\|_{\dot H^{-\eta}} \le C(1 + \|u\|_{\dot H^\kappa}^{\lceil r\rceil})\|v\|_{\dot H^{-r}}, \quad r \in \{1, \kappa - \delta\}, \quad u \in \dot H^\kappa, v \in H,$$

follow from a duality argument applied to

$$\|F'(u)v\|_{\dot H^{r}} \le C(1 + \|u\|_{\dot H^\kappa}^{\lceil r\rceil})\|v\|_{\dot H^{\eta}}, \quad r \in \{1, \kappa - \delta\}, \quad u \in \dot H^\kappa, v \in \dot H^\eta$$

in the case that $F'(u)$ is symmetric on H.

Gateaux derivative

- A mapping $F\colon U \to V$ belongs to the class $\mathcal{G}^1(U, V)$ of Gateaux differentiable mappings if, for all $u, w \in U$, the Gateaux derivative

$$F'(u)w := \lim_{\epsilon \to 0} \frac{F(u + \epsilon w) - F(u)}{\epsilon}$$

exists in V, $F'(u) \in \mathcal{L}(U, V)$ and $F'(\cdot)w\colon U \to V$ is continuous.

- If, for all $u, w_1, w_2 \in U$, it also holds that

$$F''(u)(w_1, w_2) := \lim_{\epsilon \to 0} \frac{(F'(u + \epsilon w_2) - F'(u))w_1}{\epsilon}$$

exists in V, $F''(u)\colon U \times U \to V$ is a symmetric and bounded bilinear mapping and $F''(\cdot)(w_1, w_2)\colon U \to V$ is continuous, then F is said to belong to the class $\mathcal{G}^2(U, V)$.