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UNIVERSITY OF APPLIED SCIENCES

Pair Correlations in the p-adic Setting

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Real Uniform distribution (1/2)

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$. The sequence is said to be **uniformly distributed** if for all $0 \leq a < b \leq$ it holds that

$$\lim_{N\to\infty} \frac{\#\{x_1,\ldots,x_N\} \cap [a,b)}{N} - (b-a) = 0.$$

Definition

The **discrepancy** of the first $N \in \mathbb{N}$ elements of the sequence $(x_n)_{n\in\mathbb{N}} \subset [0,1]$ is defined as

$$D_N(x_n) = \sup_{[a,b)\subset[0,1]} \left| \frac{\#\{x_1,\ldots,x_N\} \cap [a,b)}{N} - (b-a) \right|.$$

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Theorem

A sequence $(x_n)_{n\in\mathbb{N}} \subset [0,1]$ is uniformly distributed if and only if $\lim_{N\to\infty} D_N(x_n) = 0$..

Theorem (Schmidt) & Definition

The best possible rate of convergence is $D_N(x_n) = O(\log(N)/N)$. A sequence satisfying $D_N(x_n) = O(\log(N)/N)$ is called a **low-discrepancy sequence**.

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Example

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then the **Kronecker sequence** $(\{n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed. If α has in addition bounded partial quotients a_i in its continued fraction expansion, then the sequence is a low-discrepancy sequence.

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P-adic Numbers

Definition

Let $p \in \mathbb{Z}$ be a prime number. For $a = \frac{b}{c}$ with $b, c \in \mathbb{Z} \setminus \{0\}$ its p**-adic absolute value** is defined by: let m be the highest possible power with $a = p^m \frac{b'}{c'}$ and $(b'c', p) = 1$. Then

$$|a|_p := p^{-m}.$$

The **p-adic numbers** $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The **p-adic integers**

$$\mathbb{Z}_p := \left\{ x \in \mathbb{Q}_p \,:\, |x|_p \leq 1 \right\}$$

are a subring of $\mathbb{Q}_p$ and they are the closure of $\mathbb{Z}$ in the field $\mathbb{Q}_p$.

P-adic Uniform distribution (1/3)

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Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}_p$. The sequence is said to be **uniformly distributed** if for all $k \in \mathbb{N}$ and $z \in \mathbb{Z}_p$ it holds that

$$\lim_{N\to\infty} \frac{\#\{x_1, \ldots, x_N\} \cap D_p(z, p^{-k})}{N} - p^{-k} = 0,$$

where $D_p(z, p^{-k}) = \{x \in \mathbb{Z}_p \,|\, |x - z|_p \leq p^{-k}\}$.

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Example (Theorem, Cugiani/Beer)

Let $a, b \in \mathbb{Z}_p$. Then $(na+b)_{n\in\mathbb{N}}$ is a low-discrepancy sequence if and only $a \in \mathbb{Z}_p^\times = \{z \in \mathbb{Z}_p \mid |z|_p = 1\}$.

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P-adic Uniform distribution (3/3)

Theorem (W., 2024+)

Let f be a polynomial. Then $(x_n) = (f(n))$ satisfies $D_N(x_n) = \mathcal{O}\left(\frac{1}{N}\right)$ if and only if f is a permutation polynomial $\bmod p^2$.

Example

Let p be a prime. Then $(x_n) = (n^p + an + b)$ with $a \in \mathbb{Z}_p^x, a + 1 \in \mathbb{Z}_p^x$ and $b \in \mathbb{Z}_p$ is a low-discrepancy sequence in $\mathbb{Z}_p$.

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(Real) Poissonian Pair Correlations (1/2)

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$. Then its N**-point pair correlation function** is defined by

$$F_N(s) := \frac{1}{N}\#\left\{1 \leq k \neq l \leq N \ : \ \|x_k - x_l\| \leq \frac{s}{N}\right\},$$

where $\|\cdot\|$ is the distance of a number from its nearest integer and $s \geq 0$. The sequence $(x_n)_{n\in\mathbb{N}}$ has **Poissionian pair correlations** if

$$\lim_{N\to\infty} F_N(s) = 2s$$

for all $s \geq 0$.

Why do independent, uniformly distributed random sequences have PPC?

Heuristic Argument

Consider a fixed N, and fix x_n for some $1 \leq n \leq N$. Then the region around x_n with length $2s/N$ is expected to contain $2s\frac{N-1}{N}$ of the remaining points.

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$$F_N(s) \approx \frac{1}{N} \cdot N \cdot \frac{2s(N-1)}{N} \to 2s.$$

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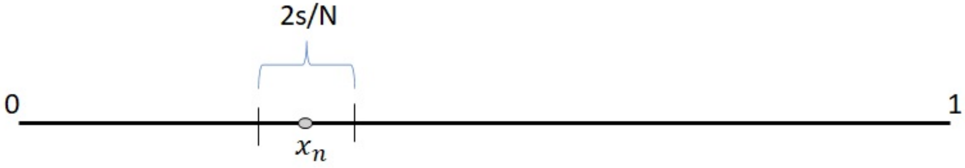

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(At least) most known examples of low-discrepancy sequence do **not** have Poissonian pair correlations, e.g. Kronecker sequences and van der Corput sequences.

Examples

The following sequences do have Poissonian pair correlations:

- $\{\sqrt{n}\}_{n \in \mathbb{N}}$ without square numbers $n \in \mathbb{N}$ (El-Baz, Marklof, Vinogradov, 2015)

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(Real) Weak Poissonian Pair Correlations

Definition

Let $0 < \alpha \leq 1$. Then a sequence $(x_n) \subset [0,1]$ has α**-weak Poissonian pair correlations** if

$$\lim_{N\to\infty} \frac{1}{N^2} \frac{1}{\mu\left(D(0, s/N^\alpha)\right)} \# \left\{ 1 \leq i \neq j \leq N \,:\, \|x_i - x_j\| \leq \frac{s}{N^\alpha} \right\} = 1$$

for all $s \geq 0$.

Proposition

Let $(x_n) \subset [0,1]$ be a low-discrepancy sequence, then (x_n) has α-weak for all $0 < \alpha < 1$.

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(P-adic) Weak Poissonian Pair Correlations (1/4)

Definition

Let $0 < \alpha \leq 1$. Then a sequence $(x_n) \subset \mathbb{Z}_p$ has **α-weak Poissonian pair correlations** if

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(P-adic) Weak Poissonian Pair Correlations (2/4)

Proposition (W., 2023)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, which are uniformly distributed on $\mathbb{Z}_p$. Then for any $0 < \alpha \leq 1$ the sequence almost surely has α-weak Poissonian pair correlations.

Theorem (W., 2024+)

Let $(x_n) = f(n)_{n\in\mathbb{N}}$ with $f(n)$ a permutation polynomial $\bmod p^2$, then the sequence has α-weak Poissonian pair correlations for any $0 < \alpha < 1$, but does not have Poissonian pair correlations.

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(P-adic) Weak Poissonian Pair Correlations (3/4)

Theorem (W., 2024+)

Let $0 < \alpha \leq 1$. If $(y_k)_{k \in \mathbb{N}} = (\varphi^{-1}(x_k))_{k \in \mathbb{N}} \in [0,1)$ has (real) α-Poissonian pair correlations, then (x_k) has (p-adic) α-Poissonian pair correlations.

Definition

The Monna map $\varphi_p : \mathbb{Z}_p \to \mathbb{R}$ is for

$$x = \sum_{i=0}^{\infty} a_i p^i$$

with $0 \leq a_i \leq p-1$, a p-adic number, defined as

$$\varphi(x) = y = \sum_{i=0}^{\infty} a_i p^{-i-1}.$$

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(P-adic) Weak Poissonian Pair Correlations (4/4)

Remark

While φ is obviously surjective, it is not injective on all of its domain due to the limit of the geometric series. Nonetheless, an inverse map φ^{-1} exists on the set

$$\left\{ x \in [0,1) \, : \, x = \sum_{i=0}^{\infty} a_i p^{-i-1}, \; a_i \neq p-1 \text{ for infinitely many } i \right\}.$$

Moreover, we know that the inverse image of x under φ contains the element

$$y = \sum_{i=0}^{\infty} a_i p^{i-1}$$

and we may thus define $\varphi^{-1}(x) := y$.

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Thank you for your attention!