Sampling recovery and sharp norm estimates of projection operators

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The plan

- relations between the Sampling, Gelfand and Kolmogorov numbers, optimality of information: **linear vs standard**;
- discretization results;
- bounds for the norms of projection operators.

Joint work with **David Krieg, Mario Ullrich and Tino Ullrich**

- *Sampling projections in the uniform norm*. arXiv: 2401.02220, 2024.
- *Sampling recovery in L_2 and other norms*. arXiv: 2305.07539, 2023.

Information $\longrightarrow$ Recovery algorithm $\longrightarrow$ Reconstructed function

$N(f)$ $\quad$ ψ $\quad$ $f \approx \psi \circ N(f)$

Information $N(f)$:

- **linear**
  - Fourier coefficients
  - **function values**
- nonlinear

Recovery algorithm ψ:

- **linear**
- nonlinear
  - **non-adaptive**
  - adaptive

$$\left\|f - \psi \circ N(f)\right\|_{\boldsymbol{G}} \to \min$$

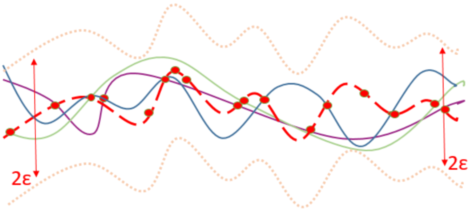
Recent overview:

D. Krieg, E. Novak, M. Ullrich, On the power of adaption and randomization. arXiv: 2406.07108, 2024.

Given: $f(x_1), ..., f(x_n)$ for functions $f \in F \subset G$.

Estimate: the **n-th linear sampling number** of F in G

$$g_n^{\mathrm{lin}}(F,G) := \inf_{\substack{x_1,\ldots,x_n \in D \\ \varphi_1,\ldots,\varphi_n \in G}} \sup_{f \in F} \left\| f - \sum_{i=1}^{n} f(x_i)\, \varphi_i \right\|_G .$$



Red line – target, other lines – approximants

Benchmark quantities

- **n-th Gelfand number** $c_n(F,G)$: **any** algorithm, **linear** information
- **n-th approximation number** $a_n(F,G)$: algorithm and information - **linear**

- **n-th Kolmogorov number**

$$d_n(F,G) := \inf_{\substack{\boldsymbol{V_n \subset G} \\ \mathbf{dim}(\boldsymbol{V_n})=\boldsymbol{n}}} \sup_{f\in F} \inf_{g\in V_n} \|f-g\|_G$$

General relations

Gelfand n.		approximation n.		sampling n.
$c_n(F,G)$	$\leq$	$a_n(F,G)$	$\leq$	$g_n^{\mathrm{lin}}(F,G)$
lin. inf.		lin. inf.		std. inf.
arb. alg.		lin. alg.		lin. alg.

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$g_n^{\mathrm{lin}}(F,G) \leq K(n) \cdot c_n(F,G)$ $\longrightarrow$ find the order of $K(n)$

For every **convex and balanced** set F, it holds

$$d_n(F, B(D)) \leq c_n(F, B(D))$$

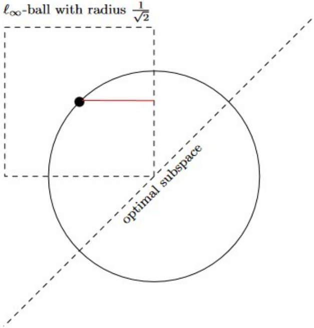
The goal: find $K(n)$ such that

$$g_n^{\mathrm{lin}}(F, B(D)) \leq K(n) \cdot d_n(F, B(D)) \leq K(n) \cdot c_n(F, B(D))$$

Kolmogorov vs Gelfand numbers

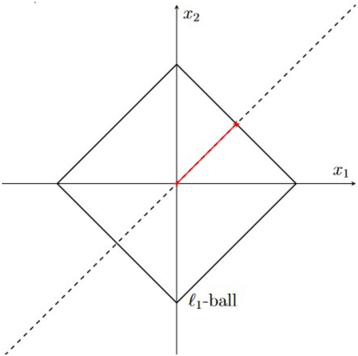
Kolmogorov numbers

$d_1(B_2, \ell_\infty)$



Gelfand numbers

$c_1(B_1, \ell_2)$



$$g_n^{\mathrm{lin}}(F, B(D)) \;\le\; K(n) \cdot d_n(F, B(D)) \;\le\; K(n) \cdot c_n(F, B(D))$$

How large are the gaps? What is the order of $\boldsymbol{K(n)}$?

Novak '88

$$g_{\boldsymbol{n}}^{\mathrm{lin}}(F, B(D)) \;\le\; \boldsymbol{(1+n)}\, d_n(F, B(D))$$

The factor $(1+n)$ cannot be improved even for Hilbert spaces.

Kashin/Konyagin/Temlyakov '21

$$g_{9n}^{\mathrm{lin}}(F, B(D)) \;\le\; 5\, d_n(F, B(D))$$

Novak ’88

$$g_{\boldsymbol{n}}^{\text{lin}}(F, B(D)) \,\leq\, \boldsymbol{(1+n)}\, d_n(F, B(D))$$

The factor $(1+n)$ cannot be improved even for Hilbert spaces.

With an oversampling bn, $b > 1$ – YES – to $\sqrt{n}$!

Kashin/Konyagin/Temlyakov ’21

$$g_{9^n}^{\text{lin}}(F, B(D)) \,\leq\, 5\, d_n(F, B(D))$$

Theorem [KPUU '23]

For **any set** D, **any** $F \subset B(D)$, and all $n \in \mathbb{N}$, it holds

$$g_{2n}^{\mathrm{lin}}(F, B(D)) \;\leq\; 58\,\sqrt{n}\; d_n(F, B(D)).$$

For **convex and balanced** F, it holds

$$g_{2n}^{\mathrm{lin}}(F, B(D)) \;\leq\; 58\,\sqrt{n}\; c_n(F, B(D)).$$

The factor $\sqrt{n}$ in general is sharp!

Note: Sharp bound without $\sqrt{n}$ for Hilbert spaces satisfying certain conditions.

Example. Sharpness of the results

Let $W_p^s := W_p^s([0,1])$, where $\|f\|_{W_p^s} := \|f\|_p + \|f^{(s)}\|_p \leq 1$, $s > 1$.

Both bounds are sharp:

$$g_n^{\mathrm{lin}}(W_1^s, B(D)) \asymp n^{-s+1}, \qquad c_n(W_1^s, B(D)) \asymp d_n(W_1^s, B(D)) \asymp n^{-s+1/2}$$

The first bound is sometimes better:

$$g_n^{\mathrm{lin}}(W_2^s, B(D)) \asymp c_n(W_2^s, B(D)) \asymp n^{-s+1/2}, \qquad d_n(W_2^s, B(D)) \asymp n^{-s}.$$

[Maiorov '76; Kashin '77; Pinkus '85; Novak '06]

Improvement in the Hilbert space setting

Let H be a RKHS on a set D with a **bounded** kernel

$$K(x,y) = \sum_{k=1}^{\infty} \sigma_k^2\, b_k(x)\overline{b_k(y)}, \tag{1}$$

where $\boldsymbol{\sigma_1 \geq \sigma_2 \geq \ldots \geq 0}$, $\boldsymbol{\{b_k\}_{k=1}^{\infty}}$ **is orthonormal in** $\boldsymbol{L_2(\mu)}$.

$$\boldsymbol{V_n = \mathrm{span}\{b_1, \ldots, b_n\}, \quad \Lambda_n^2 := \sup_{f \in V_n \setminus \{0\}} \|f\|_\infty^2 / \|f\|_2^2 = \Big\| \sum_{k=1}^{n} |b_k|^2 \Big\|_\infty}$$

Theorem [KPUU '23]

There is an absolute constant $c > 0$, such that the following holds for a RKHS $H \subset L_2(\mu)$ with the kernel (1). For all $n \in \mathbb{N}$

$$g_{cn}^{\mathrm{lin}}(H, B(D)) \leq \sqrt{\frac{866\,\Lambda_n^2}{n} \sum_{k > \lfloor n/2 \rfloor} \sigma_k^2} + \sqrt{2 \sum_{k > \lfloor n/4 \rfloor} \frac{\Lambda_{4k}^2 \sigma_k^2}{k}}.$$

Sharp bound in the Hilbert space setting

(independently obtained by [Geng/Wang '23] for the trigonometric system)

If $\Lambda_n \lesssim \sqrt{n}$, i.e.,
$$\Big\| \tfrac{1}{n} \textstyle\sum_{k=1}^{n} |b_k|^2 \Big\|_\infty \leq B, \quad B > 0,$$
then
$$g_{cn}^{\mathrm{lin}}(H, B(D))^2 \leq bB \cdot \sum_{k>n} \sigma_k^2, \qquad \sigma_k = d_k(H, L_2).$$

If μ is a **finite** measure, then

$$g_{cn}^{\mathrm{lin}}(H, B(D)) \leq \sqrt{bB \cdot \mu(D)} \cdot c_n(H, B(D))$$

by the bound $\sqrt{\sum_{k>n} \sigma_k^2} \leq \sqrt{\mu(D)} \cdot c_n(H, B(D))$ from
[Osipenko/Parfenov '95; Kuo/Wasilkowski/Wozniakowski '08; Cobos/Kühn/Sickel '16]

Linear sampling algorithms are (up to constants) **as powerful as** arbitrary non-linear algorithms using general linear measurements.

Hilbert spaces $\boldsymbol{H^w}$ on the d-torus $\mathbb{T}^d = [0, 2\pi]^d$

$$f \in L_1: \qquad \|f\|_{H^w}^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^2 |f_{\boldsymbol{k}}|^2 < \infty,$$

where $w(\boldsymbol{k}) > 0$, $\boldsymbol{k} \in \mathbb{Z}^d$, $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} (w(\boldsymbol{k}))^{-2} < \infty$,

$f_{\boldsymbol{k}}$ are the Fourier coefficients w.r.t. the trigonometric system.

If

$$w(\boldsymbol{k}) = \prod_{j=1}^{d} (1 + |k_j|)^s\,, \quad s > 1/2,$$

the orders coincide with those for sparse grids (see $[\text{Temlyakov } '93]$)

$$g_n^{\text{lin}}(H^s_{\text{mix}}, B(\mathbb{T}^d)) \asymp_{s,d} n^{-s+1/2} (\log n)^{s(d-1)}\,.$$

Nikol'skii-Besov $\mathbf{B}^r_{p,\infty} \equiv \mathbf{H}^r_p$ and Sobolev $\mathbf{W}^r_p$ mixed smoothness classes on the torus $\mathbb{T}^d$

$$\sup_{f\in F} \|f - A_m f\|_q \lesssim m^{-(r-t)}(\log m)^{(d-1)(r-t+\alpha)},$$

$r > \max\{1/2, 1/p\}, \quad t := (1/p - 1/2)_+ + (1/2 - 1/q)_+,$

$$\alpha = \begin{cases} 0 & F = \mathbf{W}^r_p, 1 \le q < \infty; \\ 1/2 & F = \mathbf{H}^r_p, 1 \le q < \infty;\ F = \mathbf{W}^r_p, q = \infty; \\ 1 & F = \mathbf{H}^r_p, q = \infty. \end{cases}$$

A_m is a plain least-squares algorithm based on subsampled random points.

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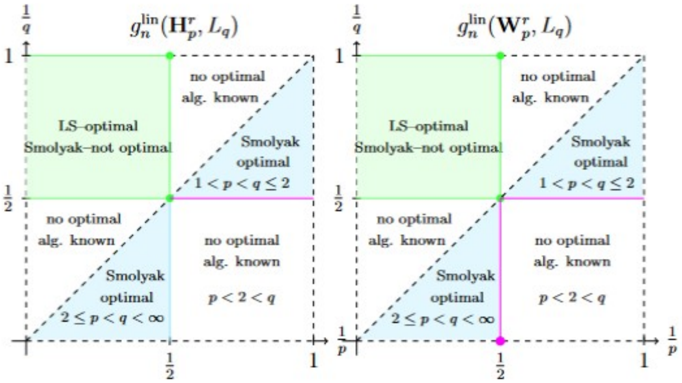
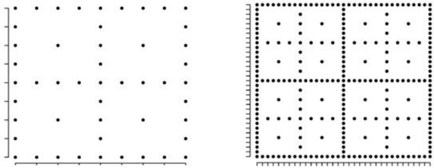
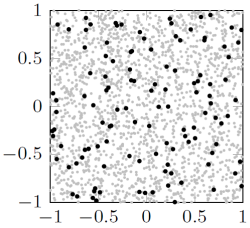


Smolyak points



Bounds for the respective worst case error using the Smolyak algorithm by [Dũng '91; Temlyakov '93; Dũng/T.Ullrich '15; / Byrenheid/T.Ullrich '17]

Subsampled random points



We also apply the results to:

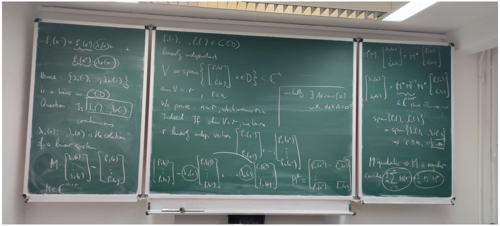
- Wiener-type and Korobov spaces on the torus $\mathbb{T}^d$
- Sobolev spaces on manifolds
- approximation in L_∞- and H^1-norm for a weigthed Sobolev space without uniformly bounded basis

Discretization of the uniform norm

Theorem [KPUU '23]

Let D be a set and $V_n \subset B(D)$, $\dim V_n = n$. Then there exist $2n$ points $x_1, \dots, x_{2n} \in D$ such that

$$\forall f \in V_n \qquad \|f\|_\infty \leq 58\sqrt{n}\Big(\frac{1}{2n}\sum_{k=1}^{2n}|f(x_k)|^2\Big)^{1/2} \leq 58\sqrt{n}\max_{k=1,\dots,2n}|f(x_k)|$$



Part of the proof.

Based on $[\text{Kiefer/Wolfowitz '60}]$ + constructive subsampling.

Algorithm for continuous on a compact domain functions

- Given n linearly independent complex-valued functions $f_1, \ldots, f_n \in C(D)$, for a fixed $x \in D$ solve

$$A := \max_{(x_k)_{k=1}^N, (\lambda_k)_{k=1}^N} \det(B^* B),$$

$$B := \begin{pmatrix} \sqrt{\lambda_1} f_1(x_1) & \sqrt{\lambda_1} f_2(x_1) & \cdots & \sqrt{\lambda_1} f_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_N} f_1(x_N) & \sqrt{\lambda_N} f_2(x_N) & \cdots & \sqrt{\lambda_N} f_n(x_N) \end{pmatrix}$$

with $N = n^2 + 1$ nodes $(x_k)_{k=1}^N$ and weights $(\lambda_k)_{k=1}^N$.

- Write the singular value decomposition $A = VDV^*$, $A^{-1/2} = VD^{-1/2}V^*$.
- Put

$$G(x) := A^{-1/2} F(x), \qquad F := (f_1, \ldots, f_n)^\top .$$

The components of $G(x) = (g_1, \ldots, g_n)^\top$ form an ONB in $V_n = \operatorname{span}\{f_1, \ldots, f_n\}$ w.r.t. the atomic measure $\varrho = \sum_{i=1}^N \lambda_i \delta_{x_i}$, and it holds

$$\Lambda_n = \sup_{f \in V_n \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_{L_2(\varrho)}} = \sup_{x \in D} \left(\sum_{i=1}^n |g_i(x)|^2 \right)^{1/2} = \sqrt{n}.$$

Discretization of the uniform norm. The best-known results.

Kashin/Temlyakov '18; Dai/Prymak/Temlyakov/Tikhonov '19

For the case of trigonometric polynomials $t \in \mathcal{T}(Q), Q \subset \mathbb{Z}^d, |Q| = n$
$\exists C_1, C_2 > 0$ such that for a set of points X, $|X| \geq C_1 n$:

$$\|t\|_\infty \leq C_2 \sqrt{n} \max_{x \in X} |t(x)|.$$

Dai/Prymak/Temlyakov/Tikhonov '19; Kashin/Konyagin/Temlyakov '21

If any $f \in V_n \subset C(D)$ satisfies the Nikol'skii inequality

$$\|f\|_\infty \leq H(n) \|f\|_2,$$

then $\exists C_1, C_2 > 0$ such that there exists a set of points $X \subset D$, $|X| \leq C_1 n$:

$$\|f\|_\infty \leq C_2 H(n) \max_{x \in X} |f(x)|.$$

Discretization of the uniform norm. The best-known results.

Kashin/Temlyakov '18; Dai/Prymak/Temlyakov/Tikhonov '19; Kashin/Konyagin/Temlyakov '21

Let $V_n \subset C(D)$. There exists a set of points X, $|X| \le 9^n$ such that for all $f \in V_n$ it holds

$$\|f\|_\infty \le 2 \max_{x \in X} |f(x)|.$$

Novak '88

Let $V_n \subset B(D)$, $\varepsilon > 0$. There exists a set of points X, $|X| = n$ such that for all $f \in V_n$ it holds

$$\|f\|_\infty \le (n + \varepsilon) \max_{x \in X} |f(x)|.$$

Theorem [KPUU '23]

Let D be a set and $V_n \subset B(D)$, $\dim V_n = n$. Then there are $2n$ points $x_i \in D$ and functions $\varphi_i \in V_n$, such that $P\colon B(D) \to V_n$ with

$$Pf = \sum_{i=1}^{2n} f(x_i)\,\varphi_i \quad \text{is a projection with} \quad \|P\| \le 58\sqrt{n}.$$

Theorem is **sharp** in the following sense:

- Using $m = \mathcal{O}(n)$ samples, the norm bound $C\sqrt{n}$ cannot be replaced with a lower-order term.
- Using only $m = n$ samples, the norm bound $C\sqrt{n}$ has to be replaced by a linear term in n (*Novak's/Auerbach's lemma*).

+ Connection to the famous Kadets-Snobar theorem (1971).

Theorem [Bartel, Kämmerer, Schäfer, P., Ullrich ’24]

Let μ be a Borel probability measure on a compact topological space D, $V_n \subset C(D)$. Then there exists an $N \leq \dim(\operatorname{span}\{f \cdot \overline{g} \colon f, g \in V_n\}) \leq n^2$ and $N+1$ points $x_1, \ldots, x_{N+1} \in D$ together with non-negative weights satisfying $\mu_1 + \cdots + \mu_{N+1} = 1$ such that for all $f \in V_n$ it holds

$$\int_D |f(x)|^2 d\mu(x) = \sum_{i=1}^{N+1} \mu_i |f(x_i)|^2 \,.$$

$\boldsymbol{L_2}$-**Marcinkiewicz-Zygmund inequalities**, see:
[Mhaskar, Narcowich, Ward ’01], [Nuyens, Cools ’06], [Keiner, Kunis, Potts ’07], [Filbir, Mhaskar ’11], [Müller-Gronbach, Novak, Ritter ’12], [Kämmerer, Potts, Volkmer ’15], [Temlyakov ’18], [Trefethen ’19], [Gröchenig ’20], [Filbir, Hielscher, Jahn, T. Ullrich ’24], ...

+ Connection to spherical designs
+ Connection to Parseval-scalable frames ...

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Thank you for your attention!