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Optimal control of joint torques using direct collocation to maximize ball carry distance in a golf swing

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Abstract Forward dynamics golf swing simulations are important to gain insight into how a golfer should swing a particular club and which design improvements should be considered by golf club manufacturers. A new method of optimizing a four degree-of-freedom (DoF) biomechanical golfer model swinging a flexible shaft with a rigid clubhead was developed using a direct orthogonal collocation approach. The kinematic and kinetic results of the simulation confirm previous findings on optimal joint angle trajectories, shaft deflection patterns, and joint torque profiles in a golf swing. This optimization approach is a promising development in biomechanics research, and future work will implement this method in three-dimensional swing models that have been shown to have higher robustness and fidelity.

Keywords Golf swing · Forward dynamics · Optimal control · Direct collocation

1 Introduction

The human body can be described as a multibody system, and multibody biomechanics is an evolved area of research in sports engineering and rehabilitation, particularly in golf. Golf was recently cited by National Allied Golf Associations to have an economic impact of over $19 billion per year in Canada alone [1]. Research in improving golf club design is of great interest to the industry, as well as improving biomechanical models of golf swings. By modeling the golf swing, insights can be made into how the golfer should swing a particular club and which improvements should be considered in golf club design.

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Over the last 40 years, models of the golf swing have been developed for use in inverse dynamic simulations, which require input experimental data, and forward dynamic simulations, which require the input of unoptimized or optimized torques [2]. One of the first golf swing models produced to gain insight into the biomechanics of the swing was the double pendulum by Cochran and Stobbs [3], in which two links represented the arms and the club. This study found that a well-coordinated swing could be produced with a passive wrist joint, as the club could be accelerated naturally by the centrifugal force acting on its centre of gravity. The validity of the double pendulum model was verified in a comprehensive forward dynamic analysis by Jorgensen [4], in which the Lagrangian method was used to generate the differential equations of motion. Jorgensen generated solutions to these differential equations by inputing a constant torque at the shoulder and wrist and also found that a passive or low active torque produced a natural downswing with high clubhead velocity. This effect was also confirmed in a forward dynamic study of the double pendulum model by Lampsa, in which joint torque inputs were determined using Pontryagin's Maximum Principle [5]. This was one of the first golf swing analysis studies to produce a realistic golf swing simulation using optimized joint torques.

More recently, golf swing models have evolved to include biomechanical elements and multiple degrees of freedom for use in forward dynamic simulations. Sprigings and Neal developed a planar three segment model that included a rotating torso [6], which was later expanded by Mackenzie and Sprigings [7] into a three-dimensional four DOF model that included forearm rotation. This model, along with other recent forward dynamic golf simulations, used a parameter identification method to optimize the active torque produced by each joint in the golfer [7,8,9]. This was similar to recent forward dynamic simulations in other sport movement applications, including tumbling, jumping, and tennis [10,11, 12]. A continuous function represented by Eqn. 1 was used to mimic the maximum isometric activation joint torque, independent of joint angle, throughout the swing [6]

$$T_{pre}(t) = T_m\left(1 - e^{\frac{t_{on}}{t_{act}}}\right) - T_m\left(1 - e^{\frac{t_{off}}{t_{deact}}}\right) \quad (1)$$

where an optimal activation torque, $T_{pre}(t)$, is produced by selecting optimal activation time, t_{on}, and deactivation time, t_{off}, and is limited by the maximum torque producing capability, T_m, activation constant, t_{act}, and deactivation constant, t_{deact}, of muscles. This method of determining optimal joint torques has shown to be effective in producing similar results to actual golf swings. However, it is an approximation limited by the shape of Eqn. 1 and a single, maximized activation and deactivation of the joint. It is of interest to compare this with the optimal control generated without a restriction on the threshold and frequency of activation that drives the joints in the model. As with any advanced sport movement, the golf swing is susceptible to large performance differences given small variation in activation timings. This proves to be one of the key factors in the difficulty of this particular optimization problem, so it is necessary to improve the optimization methods used in forward dynamics golf swing simulations. An alternative optimization approach that doesn't require of optimization of single activation parameters is a direct-collocation optimal control method, in this case implemented by GPOPS-II [13].

Direct-collocation can be taken advantage of to generate activation profiles for each torque generator that are not represented by a pre-defined function (ie. Eqn. 1). A previous forward dynamic model for Paralympic curling utilized direct collocation (in GPOPS-II) to obtain an efficient method of optimizing a dynamic biomechanical joint torque model [14]. In other biomechanical applications, direct collocation was found to be a promising improvement over other optimal control methods by increasing the computational efficiency of dynamic optimization [15]. However, direct collocation has yet to be used in a forward dynamic golf swing simulation.

2 Methods

2.1 Golfer Biomechanical Model

In recent studies by Mackenzie and Sprigings [7] and Balzerson et al. [8], a four degree-of-freedom (DoF) golfer model incorporating a two DoF flexible club was shown to provide close comparison to experimental kinematics and kinetics of a low handicap golfer. The same model template was used in this study. A global inertial frame was defined as a basis for the model's motion, with representation for the downrange target direction (X), the vertical (Y), and the perpendicular to the golfers stance (Z, following the right-hand-rule convention). The four DoF of the golfer were provided by four revolute joints to allow torso rotation, transverse flexion and adduction of the arm, supination and pronation of the forearm (alternatively, could represent the internal-external rotation of the shoulder), and ulnar and radial deviation of the wrist. This model is represented in Figure 1. The human biomechanical model included four rigid body segments representative of the golfer's single upper arm, forearm, hand, and torso. To provide a realistic golfer stance, the torso segment was constrained to rotate about an axis 35° to the vertical, in which a zero degree rotation was defined as the approximate address position of the golfer. The swing plane of the arm was rotated 70° from the horizontal [7], which allowed the arm to sweep downwards towards the ball during impact. A zero degree angle of the shoulder approximated the address position of the golfer. The distance from the proximal torso to the shoulder joint was set to 20 cm, which provided a representation of the half shoulder width of the golfer. The golfer's elbow was fixed at 180°, allowing the golfer to swing with a straight arm, in which a 90° angle of the forearm represented the approximate address position of the golfer. Furthermore, a zero degree angle approximated the neutral position of the wrist, which was the assumed position at the top of the backswing.

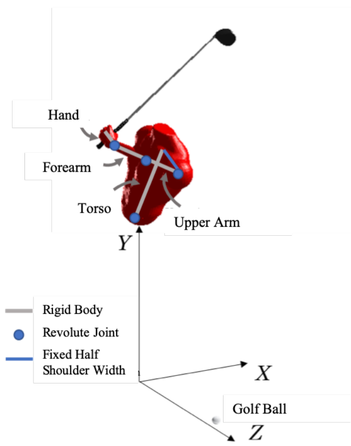


Fig. 1 Four DOF golfer model relative to the inertial coordinate system at the initial condition

De Leva obtained body segment inertial parameters and joint centre end point locations for college-aged Caucasian males and females using cadaver data [5]. De Leva's body segment data can be extrapolated based on subject height and mass. In this study, an average subject height and mass of 1.74 m and 73.0 kg were used for scaling of the body segment parameters. The resulting parameters for each segment used in this model are displayed in Tab. 1, and are similar to those used by Balzerson et al. and Mackenzie and Sprigings.

Table 1 Biomechanical parameters for the golfer taken from De Leva's body segment data [16]

Segment	Length (m)	Mass (kg)	Center of mass (m)	Inertia$_{xx}$ (kgm^2)	Inertia$_{yy}$ (kgm^2)	Inertia$_{zz}$ (kgm^2)
Upper Trunk	0.171	11.7	0.0510	0.0700	0.147	0.174
Mid Trunk	0.215	11.9	0.0970	0.0810	0.121	0.129
Lower Trunk	0.146	8.15	0.0890	0.0530	0.0600	0.0650
Upper Arm	0.282	1.98	0.163	0.00400	0.0110	0.0130
Forearm	0.269	1.18	0.123	0.00100	0.00600	0.00700
Hand	0.0860	0.450	0.0680	0.00100	0.00100	0.00100

2.2 Flexible Golf Club Model

The golf club consists of a flexible shaft rigidly fixed to a clubhead at one end, and rigidly fixed to the center of mass of the hand segment at the other. The flexible shaft is modeled as complete second-order elastic rotation matrix for a Rayleigh beam and is available as a component in MapleSim. The dynamic equations are validated in the work of Sandhu et al. and a detailed description can be found in [17]. This flexible shaft model provides two DOF to the clubhead, deflection in both transverse directions. The flexible beam component is able to model the dynamics of the shaft by allowing for the moment of inertias, cross-sectional area, and Young's modulus to be described as sixth-order polynomial functions of shaft distance from the bottom of the grip. These polynomials were obtained by fitting to manufacturer provided data [18]. The clubhead is modeled as a rigid body fixed to the end of the flexible beam. The inertial properties for the clubhead are taken from those determined by [18]. As suggested by [18], the Young's modulus values were scaled by a factor of 1.5 to obtain more realistic maximum deflections.

2.3 Optimal Control Design

2.3.1 Optimization Method

Direct collocation was used to solve the optimization problem of this study. In this method, the state and control are approximated over subintervals by nth-degree polynomials. The continuous-time optimal control problem is then transcribed to a finite Nonlinear Programming Problem (NLP), where the NLP can be solved using well-known software, such as IPOPT. This solution is obtained iteratively through mesh refinement methods, which both modify the number of subintervals and nth-degree polynomials, until the objective function is minimized with satisfied constraints [19]. GPOPS-II is a general-purpose MATLAB software program for optimal control problems that utilizes the Legendre-Gauss-Radau (LGR) orthogonal direct collocation method. LGR orthogonal collocation is defined by the method that collocation points are chosen, where collocation points are selected to be the roots of a Legendre polynomial [13]. The IPOPT NLP software package was utilized to solve the resulting large-scale NLPs with a relative tolerance of 10^{-10} and maximum iterations of 20000. Furthermore, mesh iterations were continued until a mesh tolerance of 10^{-5} was reached.

In this optimal control problem, the aim was to determine the optimal input function $u^*(t)$ for the dynamic system

$$\dot{x}(t) = f[x^*(t), u^*(t), t], \tag{2}$$

that was subject to the boundary conditions

$$\phi_{min} \leq \phi[x(t_0), u(t_0), t_0, x(t_f), u(t_f), t_f] \leq \phi_{max} \tag{3}$$

where t_0 represents the initial time, t_f represents the final time, and ϕ represents the generalized constraints to the states, control, and time of the optimal control problem, and that allowed an optimal state trajectory $x^*(t)$ to minimize the objective function

$$J = \Phi[x^*(t_0), t_0, x^*(t_f), t_f)] + \int_{t_0}^{t_f} \mathcal{L}[x^*(t), u^*(t), t]dt, \tag{4}$$

Eqns. 2 to 4 form the foundation of the optimal control problem, and will be further defined for this application in the follow subsections.

2.3.2 State and Control

MapleSim was used to obtain the differential dynamic equations described by Eqn. 2 in the form of an ordinary differential equation (ODE) for each state. To reduce the complexity of the problem, the flexible beam component in MapleSim only allow transverse deflection in the swing and droop plane directions. The magnitude of torsion is minimal compared to the bending deflection of the club, and therefore was not included in this model [7]. The states were written as the generalized coordinates of the biomechanical golfer model and flexible shaft,

$$x(t) = \left[\theta_B(t), \dot{\theta_B}(t), v_S(t), \dot{v_S}(t)\right] \tag{5}$$

where the states $\theta_B(t), \dot{\theta_B}(t)$ represent the angle and angular velocity of the torso, shoulder, forearm, and wrist, and $v_S(t), \dot{v_S}(t)$ represents the bending coordinates of the shaft, with one in each transverse direction. The inputs to the system to be optimized were generated torques at the torso, shoulder, forearm, and wrist joint,

$$u(t) = [T_T(t), T_S(t), T_F(t), T_W(t)]\,. \tag{6}$$

Generated torques were scaled by the Hill force-velocity relationship of muscle, shown by Eqn. 7, and first implemented in a golf swing model by [6],

$$T_{V_c}(t, \omega) = T_{pre}(t)\frac{\omega_{max} - \omega}{\omega_{max} + \Gamma\omega} \tag{7}$$

where the net active torque of each joint, $T_{V_c}(t, \omega)$, is produced by scaling the activation torque, $T_{pre}(t)$, with the maximum possible joint angular velocity, ω_{max}, instantaneous angular velocity, ω, and scaling constant, Γ.

2.3.3 Bounds

The boundary conditions that were used to describe Eqn. 3 are summarized in this subsection. The initial joint angles used by [7] were selected for this study, which corresponded to 0°, -70°, 0°, and -35° for the torso, shoulder, arm, and wrist, respectively. Although [7] used 0° for the initial wrist angle, a relative angle of 35° was input between between the club and hand in this study to provide a more realistic hand-club grip [9]. Therefore, the initial wrist angle of -35° was required to produce the same club-forearm angle as [7] of 0°. Furthermore, generous bounds were placed on each joint angle to represent their respective maximum and minimum range of motion.

In addition, joint torques were bound by the constraints used by Balzerson et al., which are listed in Tab. 2 [8]. The optimizer selected optimal torque activations between 0 and 10% of the respective maximum value for each joint to represent an initial activation of muscle. An initial angular velocity of -2 rad/s was input to the shoulder to simulate the final moment of the backswing, which typically generates an initial shaft deflection at the top of the downswing. To ensure a realistic initial downswing position, the shoulder angle was limited to reach -75°. Additionally, rate of torque development (RTD) is the phenomenon in which muscles are limited by how quickly they can reach maximum force [20]. To account for this, it was necessary to bound the lower and upper limits of the rate of change of the shoulder and wrist torque. In previous studies, mass normalized RTD was found to vary proportionally with joint velocity in the range of 40-130 Nm/kg/s in an upper limb torque interaction study by Bastian et al. [20]. Taking the golfer's mass used in this simulation into account, the shoulder RTD was bounded at ±3200 Nm/s. Although wrist RTD values were not provided in Bastians study or in other RTD literature, they were assumed to be in proportion to the torque activation function used by Mackenzie and Sprigings and Balzerson et al. and are found in Tab. 2.

Table 2 Parameters for the torque generated at each joint

Joint	Bound (Nm)	ω_{max} (rad/s)	Γ	RTD (Nms^{-1})
Torso	200	30	3.0	4000
Shoulder	160	30	3.0	3200
Forearm	90	60	3.0	1800
Wrist	90	60	3.0	1800

2.3.4 Objective Function

The summarize the third component of the optimal control problem, the objective function defined by Eqn. 4 is further described here. Maximum ball carry has been used in previous optimization studies to obtain an optimal set of input joint torque activations [8, 18]. An impulse-momentum model tuned for clubhead-ball impacts was used to calculate ball launch conditions from clubhead kinematics, in which a center face impact was assumed [8]. The ball launch conditions were then input to a golf ball aerodynamic model for calculating ball flight dynamics [21]. Bounds were placed on the possible impact location between the center of the clubface and golf ball to ensure a realistic ball strike. The model was allowed to impact the ball between the ground (0 m) and the maximum allowable tee height (0.1 m). Furthermore, the ball could be struck between a position behind center stance (-0.05 m) and 0.4m downrange, as well as between 0.8-2.0 m perpendicular to the address position, which provided a realistic range of ball positions that could be obtained by the golfer. To assist in minimizing ball carry deviation from center, the maximum allowable side spin of the ball at launch was ±300 rpm. To achieve maximum downrange ball carry, the resulting objective function that was minimized, in the form of Eqn. 4, can be displayed as

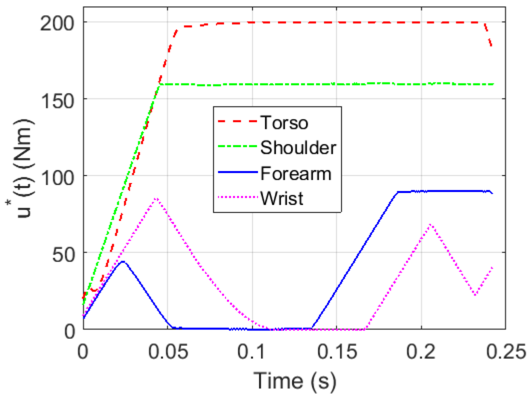
$$J = \frac{1}{X^2} \tag{8}$$

where X represents final downrange carry. With this simple objective function containing only an end state variable, the integral term in Eqn. 4 is not needed. As with [18], lateral ball deviation was not minimized to allow freedom of the model to choose optimal alignment.

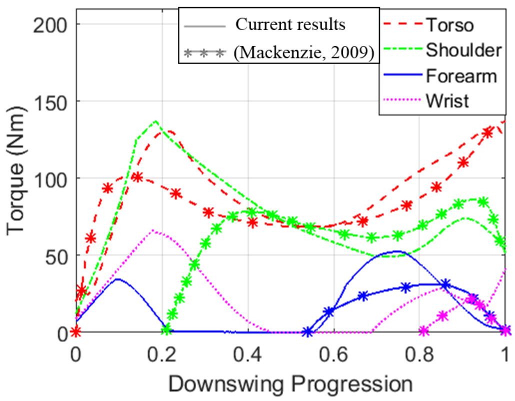
3 Results and Discussion

3.1 Optimal States and Control

Figure 2 shows the resulting optimal activation torque and velocity scaled torque produced by each joint as a result of force-velocity scaling. To maximize the swing velocity, it was expected that optimal control of each joint would attempt to reach its maximum torque value at some point during the swing. All joints activate at the onset of the swing to transition from backswing to downswing. During the downswing, the torso and shoulder utilize their maximum rate of



(a) Activation torque



(b) Velocity scaled input torque

Fig. 2 Optimal activation torque and scaled input torque generated by the golfer model for the results of this study and Mackenzie and Sprigings [7]

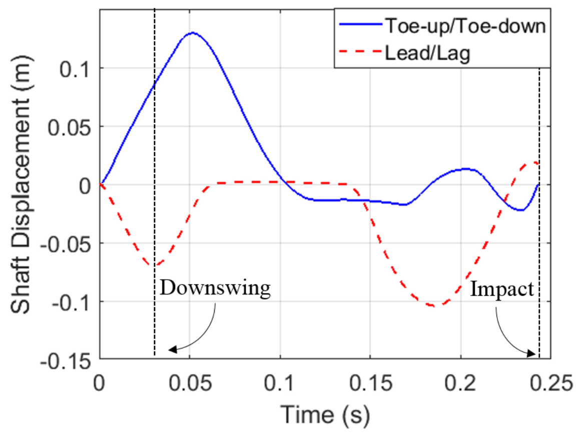


Fig. 3 Displacement of the clubhead relative to the grip of the shaft in the swing plane

development to reach peak activation torque for the majority of the swing. After the initial stabilizing torque during the swing transition, the forearm activates later in the downswing, which is followed by activation of the wrist. This activation pattern is the same as that observed in the results of Mackenzie and Sprigings [7]. By utilizing the same torque-velocity function as [7], a close comparison in net torque profile can be observed between the results of [7] and this study. This is particularly found in the torso and shoulder during maximal activation. Furthermore, the optimizer of this study chose very similar activation times for both the forearm and wrist to that of [7]. The benefit of using the multiple activation method of this study was the ability to provide stabilization of the club during the transition of backswing to downswing.

In Figure 3, the profile of the shaft deflection is similar to that of the Mackenzie and Sprigings model and experimental determined shaft deflection plots in the literature [4,9,7]. Mackenzie and Sprigings state that the lead/lag deflection should reach its maximum value after the maximum toe-up deflection, and that the shaft should have positive lead at impact [7]. One deviation of the results of this study is the neutral toe-up deflection at impact that was determined by the optimizer, whereas a small toe-down deflection was expected. This may have been the result of the relatively steep shoulder plane that was selected for the biomechanical model. The shoulder plane was selected to both match the initial grip kinematics to experimental values, as well as to maximize ball carry. It was found through simulation trials that a small shoulder plane reduced error in initial grip kinematics, and larger shoulder plane angles resulted in farther ball carry distances. The benefit of the model of [9] is the variable shoulder plane angle that results from a two-DOF shoulder joint, which gives further control to the model to identify an optimal and realistic downswing. To minimize the oscillatory shaft vibration observed in the results of Balzerson et al. [8], a shaft damping value of 6×10^{-3}s was used, which was identified in a previous study [9].

3.2 Objective Function Solutions

The optimal control was solved through minimization of the performance index from Eqn. 8. The optimal control and state trajectories produced a final clubhead speed of 46.7 m/s (105 mph) and ball speed of 68.6 m/s (153 mph) in a downswing time of 0.243 s. The clubhead struck the ball at an optimal ball height of 10 cm above the ground (the maximum allowable tee height), and 7.09 cm downrange of the center of stance. These results align with expectations, as the optimizer found an ideal clubhead delivery by maximizing tee height to increase the angle of attack and dynamic loft, similar to results obtained in [9] in which the same tee height was determined. Furthermore, most golfers tend to place the ball further ahead in their stance when using the driver to maximize ball carry, compared to a ball position closer to middle stance when using irons. The angle of attack of the club was 3.44°, and dynamic loft was 2.37°. This resulted in a vertical launch angle of 9.42° with a back spin of 2250 rpm. The optimizer selected a horizontal launch angle of 3.13° (push to the right) and draw spin of 300 rpm to maximize the downrange carry of the ball. It is likely that the biomechanical constraints of the model didn't allow for the optimal shot to follow a perfectly straight path. The smash factor, which is known as the ratio of ball speed to clubhead speed, was quantified by [9] to be 1.47 from simulation results, and 1.48 on the PGA Tour. Although [9] used a volumetric contact model, the smash factor of this study was found to be very similar at 1.46. The ball flight profile can be observed in Fig. 4.

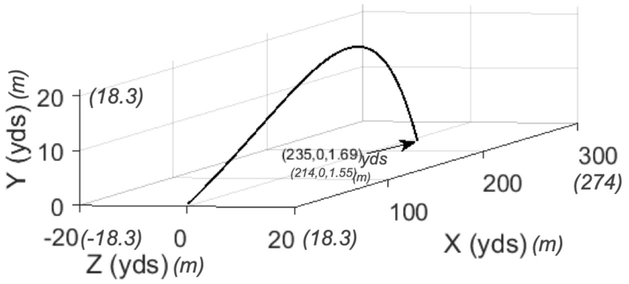
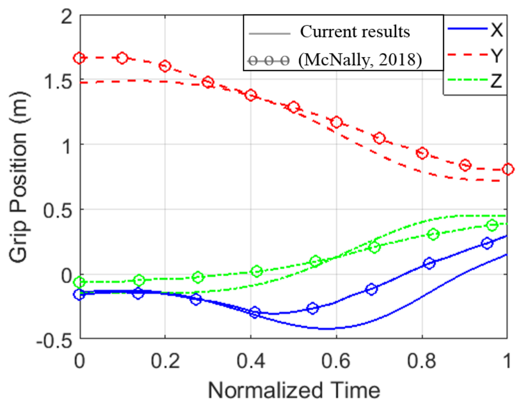
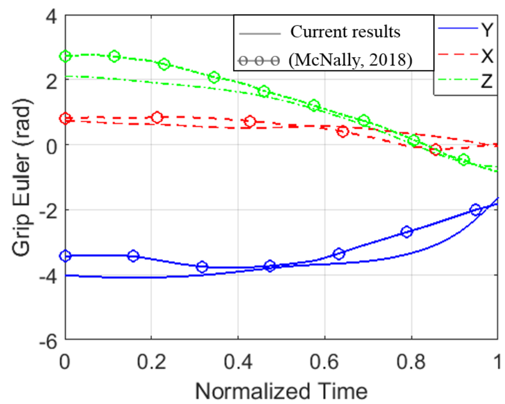


Fig. 4 Simulated ball flight



(a) Grip Position (Inertial)



(b) Grip Orientation (Euler Y-X-Z)

Fig. 5 Grip kinematics between this simulation and mean experimental results from [18]

The swing kinematics were further validated by the comparison of grip position and orientation to the mean results of 100 experimental golf swings presented by [18], which is found in Fig. 5. The simulated kinematics show a similar profile to the mean experimental results. Small discrepancies can be seen in the initial simulated grip kinematics, which are partly a result of the biomechanical constraints of the model, as well as the small initial backswing of this study.

4 Conclusion

In summary, a golf swing model including a flexible club and four degrees of biomechanical freedom was created in MapleSim, and the resulting differential equations were exported to GPOPS-II for optimization of downrange ball carry. Torque scaling by a muscle force-velocity relationship was included in the dynamic model, and a limit on the rate of torque development was implemented to impose realistic biomechanical constraints on the optimal solution. Performing the optimization using direct collocation provided the opportunity to observe how an optimized torque profile would compare to an optimized pre-defined single activation function. Direct orthogonal collocation allows for a more robust optimization method, as it identifies a unique, continuous time series input rather than identifying optimal timing parameters for a pre-defined input function. This optimization method was applied to a golfer model with state-of-the-art sub-models, including a 3-dimensional swing with biomechanical muscle scaling, use of validated flexible beam model, aerodynamic ball flight, and impact model to maximize downrange ball carry distance. This study produced optimized torques with one to two primary activation sequences in each joint, with the order of these activations occurring in similar timing to those determined in literature. Overall, this study found that the direct orthogonal collocation method can successfully optimize a forward dynamic golf swing model. Future work should aim to utilize this robust optimization method to advance biomechanical golfer model research, such as the model developed by [9].

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