Stochastic Renewal Processes in Structural Reliability Analysis:

An Overview of Models and Applications

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Context

- Structural reliability theory is well established as a major tool for
  - assessing the safety of infrastructure systems
  - guiding investment decisions for renewal of ageing infrastructure
- The time-dependent reliability analysis is founded on the theory of stochastic processes
  - Modeling of environmental/traffic loads, degradation phenomena, and cost resulting from accidents and failures
- This presentation provides a more unified view of stochastic models that are commonly used in reliability practise
  - Applications of stochastic processes are summarized in terms of two key technical problems
  - More refined models are developed

Two Main Streams of Engineering Reliability Analysis

Reliability of Structural components Started in 1940s and 50s with aircraft structures (Pugsley) and fatigue analysis (Freudenthal)

Reliability of electronic components Started in 1950s with radio equipment (Barlow and Proschan's book in 1965: Mathematical Theory of Reliability)

Two types of Failure Modes

1. Non-repairable failures (structural problems)
2. Repairable failures (equipment reliability)

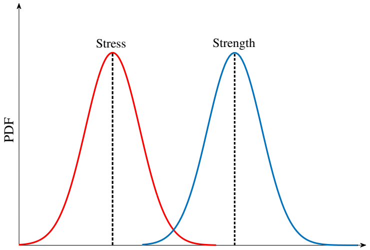
Non-Repairable Failure

Large damage may require full replacement

Repairable Failure

Components can be restored to the operating condition

Basic Structural Reliability Problem



- The stress and strength are random variables
- A Familiar limit state expression for the Probability of failure
- $P_f = \mathbb{P}\left[g(\mathbf{X}) \leq 0\right],$ where $g(\mathbf{X})$ is the limit state function
- Computation of P_f for more general functionals of stress and strength (FEM method etc) - An active area of research

Variation over Time

What is the nature of variation over time?

- Repetitive stress events (accidents)
- Loss of strength over time (in a random manner)
- A combination of these two problems

Analysis of two failure modes

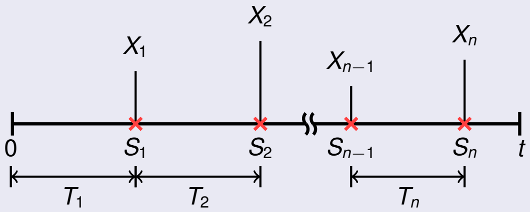
(1) Repairable, and (2) Non-repairable modes

Move toward the theory of stochastic processes

- To model time-dependent events in reliability analysis
- A stochastic process is essentially a vector of random variables indexed with time, and with a dependence structure

Structural Reliability Analysis

Conceptual Modelling of a Recurring Hazard



- Random arrival of hazards at times, $S_1, S_2, \ldots$
- Each load event has an attribute given by RVs, $X_1, X_2, \ldots$
- Load duration can be negligible (not always)
- An independent sequence of vectors, $(T_i, X_i), i = 1, 2, \ldots$

Two Key Problems of Interest

(1) Maximum Value

The distribution of maximum value generated by $N(t)$ (random) occurrences of the process

$$\mathbb{P}\left[X_{max}(t) \leq x\right] = \mathbb{P}\left[X_1 \leq x, X_2 \leq x, \ldots, X_{N(t)} \leq x\right]$$

(2) Cumulative Effect

The distribution of total effect, i.e., a (random) sum:

$$Y(t) = \sum_{1}^{N(t)} X_1$$

Maximum Value

Simplifying a time-dependent reliability problem

- Find the distribution of the maximum load, $X_{max}(t)$, in an interval $(0, t]$, e.g., design life
- Compute the probability of failure in the same way as in a static reliability problem

$$P_f(t) = \mathbb{P}\left[X_{max}(t) > R\right]$$

- This will give the probability of first failure in a non-repairable problem
- Design codes are based on this line of thinking

Cumulative Effect

Ubiquitous in life-cycle analysis models

- Degradation modelling and reliability prediction over an interval, $(0, t]$
- Total (cumulative) cost of damage due to recurring hazards
- Inspection and maintenance planning

Analytical Tools

- Most common tool is the method of moments
- Expected value of the cumulative sum is easily calculated
  Occasionally, variance is also computed

The Most Popular Model: The Poisson Process

- The seismic risk analysis led the way by introducing the Poisson process model to solve the two problems (Cornell, 1968)
- The analytical formulation is remarkably simple

The distribution of maximum value

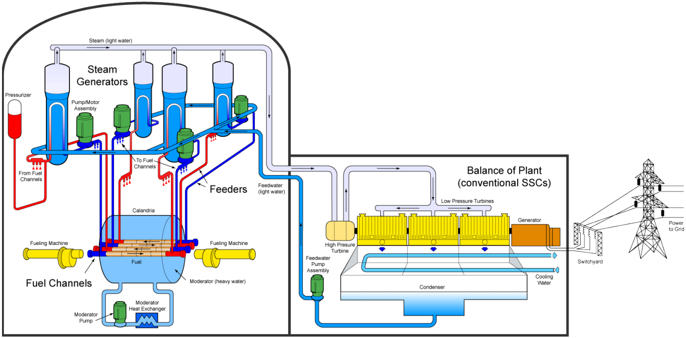
$$F_{X_{max}}(x;t) = e^{-\lambda t \overline{F}_X(x)}.$$

Expected cumulative effect

$$\mathbb{E}\left[Y(t)\right] = \lambda\, t\, \mathbb{E}\left[X\right]$$

Many solutions have been formulated using these results

New Class of Problems: Ageing Fleet of Reactors



(image courtesy of AECL)

- Many nuclear reactors in Canada and elsewhere are nearing the end of life - Refurbishment projects are huge investment
- How to ensure safe, reliable and economical operation of aged facilities over the remaining life?

The Need for More Refined Models

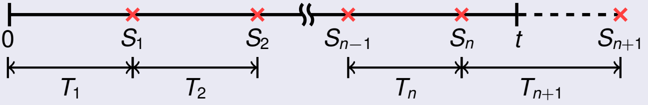
- The remaining life is relatively short, such that simple asymptotic solutions are not applicable
  - Solutions are required for a finite service life
- Assumptions underlying the Poisson model are not applicable to ageing systems and some hazards
  - Exponential inter-arrival time and lack of memory property
- Accurate estimates of expected cost and variance of the cost are required for a credible risk assessment
  - Probability distribution and various percentiles are required for decision making
- More fundamental understanding of stochastic models is required to solve these problems

Outline

- Stochastic processes in structural reliability analysis
  - Renewal Process
  - Marked Renewal Process
  - Compound Process
  - Alternating Renewal Process
- Application to problems related to reliability analysis and life-cycle assessment

Renewal Process - Random Arrivals of Events

A schematic



- A special case of the stochastic *point process*
- A sequence of inter-arrival times, $T_1, T_2, \cdots,$ are *iid* random variables with a common distribution, $F_T(t)$
- The arrival time of an n^{th} event, S_n, given as a partial sum: $S_n = T_1 + T_2 \cdots + T_n$
- The number of events, $N(t)$, in $(0, t]$ is referred to as a *counting process* associated with the partial sum, $S_n, \; n \geq 1$

Elements of Probabilistic Analysis

- The most significant quantity is the joint distribution of arrival times, $S_1, S_2, \dots S_n$
- The marginal distribution of an n^{th} arrival time, S_n
- The probability distribution of the number of events, $N(t)$

Key Analytical Issue

- How to specify the dependence structure in the joint distribution of arrival times?
- the distribution of S_n involves a convolution of n RVs
- The distribution of $N(t)$ is related to that of S_n as

$$\mathbb{P}[N(t) = n] = \mathbb{P}[S_n \leq t < S_{n+1}] \tag{1}$$

Renewal Function

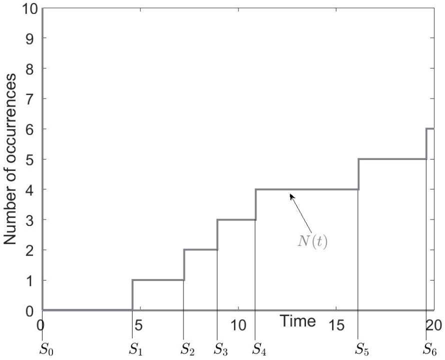
- It is a key descriptor of the renewal process
- It is defined as the expected number of events in $(0, t]$

$$\mathbb{E}\left[N(t)\right] = \Lambda(t) = \sum_{i=1}^{\infty} \mathbb{P}\left[S_i \leq t\right] = \sum_{i=1}^{\infty} F_T^{(n)}(t)$$

- It is a sum of high-order convolutions!
- A direct computation by the first principle is not possible
- The two factors are helpful to solve this problem
  - The inter-arrival times, $T_1, T_2, \ldots T_n$ are *iid* random variables
  - The regenerative property of the renewal process

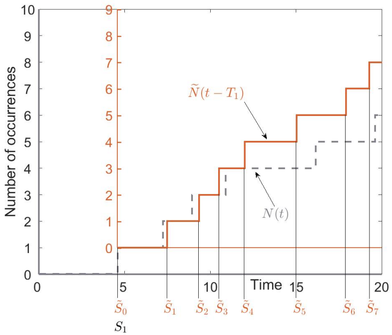
Regenerative Property of the Renewal Process

A Counting Process $N(t)$



Regenerative Property (2)

Consider another counting Process $\widetilde{N}(t)$ starting at time S_1



- The shifted process, $\widetilde{N}(t)$, is probabilistically identical to the original process, $N(t)$
- Identical in distribution and moments

Renewal Equation

Renewal integral equation

$$\Lambda(t) = F_T(t) + \int_0^t \Lambda(t-x)\,\mathrm{d}F_T(x).$$

- It is a recursive equation in time
- Analytical solution in some special cases (Erlang-2 distribution)
- Numerical computation by the trapezoidal integration rule
- A simple algorithm for computation is available!

Poisson Process Model: A Special Case

- Why we don't see the renewal equation so frequently?

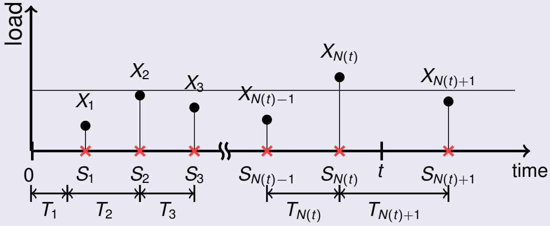
Features

- Inter-arrival time (T) has Exponential distribution
- The number of events, $N(t)$, has Poisson distribution
- The joint distribution of arrival times, $S_1, S_2, \ldots, S_n$, is known
- The marginal of S_n (n^{th} arrival time) has Gamma distribution
- All the moments are known

- The linear form of the renewal function, $\mathbb{E}[N(t)] = \lambda t$
- A constant renewal rate, λ
- In a technical sense, problems related to the Poisson process admit relatively simple solutions

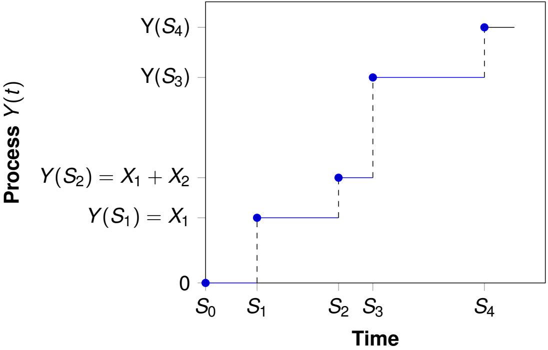
Marked Renewal Process

Conceptual model



- Arrival of events (or shocks) as a renewal process
- Each arrival is accompanied by a (random mark), $X_1, X_2, \ldots$
- T_i and X_i are typically independent
- $(T_1, X_1), \ldots, (T_n, X_n)$ are *iid* vectors

Compound Renewal Process

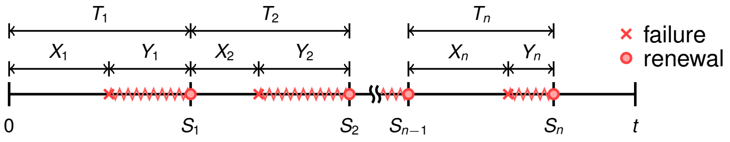


- Evaluation of the total effect

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

- X_i can be degradation caused by an i^{th} shock
- X_i can be cost of damage resulting from an i^{th} shock

Alternating Renewal Process



- An important problem for safety systems (in nuclear and processing industry)
- The system is operating for a (random) time X, and then it is down for time Y for repair, followed by a failure
- The probability that system is unavailable at time t
- This process is also relevant to probabilistic modelling of resilience problem

Applications of the Renewal Process

Maximum value problem

- Distribution of maximum load generated by a renewal process, and alternating process

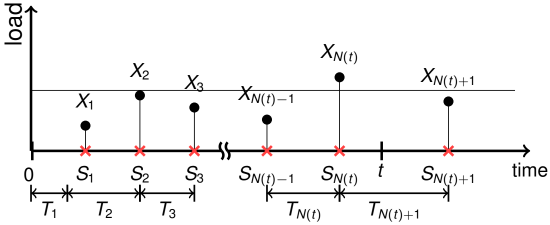
Cumulative Process

- Total damage cost analysis (over the life cycle):
  - Mean and variance of the discounted life cycle cost
  - Distribution of the total cost
- Degradation process models
  - Gamma process, compound process, shock models

Combination problems

- Load combination problem (Shock and pulse load processes)
- Stochastic shock and degradation processes

The Distribution of Maximum Load



- The maximum load in an interval $(0, t]$

$$X^*(t) = \max(X_1, \ldots, X_{N(t)})$$

- From independence of loads,

$$\mathbb{P}(X^*(t) \leq x) = \mathbb{E}\left[F_X(x)^{N(t)}\right]$$

- The maximum is an expectation w.r.t. the counting process

Solution

- This is case is not as straightforward as the Poisson process
- The distribution of $N(t)$ is not explicit
- The marked process is regenerative at time T_1

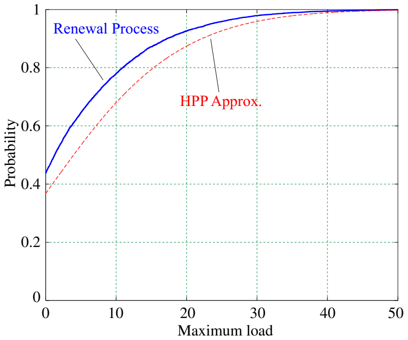
$$X(t) = X_1 \mathbf{1}_{\{T_1 = t\}} + \widetilde{X}(t - T_1), \quad (T_1 \leq t)$$

- Processes $X(t)$ and $\widetilde{X}(t)$ have same probabilistic structure

Final Expression - An integral equation

$$\mathbb{P}(X^*(t) \leq x) = \overline{F}_T(t) + F_X(x) \int_0^t \mathbb{P}(X^*(t-u) \leq x) \, \mathrm{d}F_T(u)$$

Example



- The distribution of maximum value for two different time horizons (10 years and 30 years)
  - T is lognormal and X is Pareto distributed with suitable parameters
- The Poisson process approximation is also compared

Distribution of the Maximum Load: Generalization

- A unified nature of the solution for other two cases
  - Renewal pulse process
  - Alternating renewal process

Process	Formula
Renewal process	$\mathbb{P}(X^*(t) \leq x) = \mathbb{E}\left[F_X(x)^{N(t)}\right]$
Pulse process	$\mathbb{P}(X^*(t) \leq x) = F_X(x)\,\mathbb{E}\left[F_X(x)^{N(t)}\right]$
Alternating process	$\mathbb{P}(X^*(t) \leq x) = F_X(x)\,\mathbb{E}\left[F_X(x)^{N(t)}\right]$

- Useful for risk assessment of systems nearing the end of life

Total Damage Cost over the Life Cycle

- The total damage cost, $K(t)$, is a compound renewal process

$$K(t) = \sum_{i=1}^{N(t)} C_i$$

- $C_1, C_2, \ldots$: (random) damage costs caused by different events
- The total discounted cost

$$K_D(t) = \sum_{i=1}^{N(t)} C_i \, e^{-\rho S_i}$$

- Moments of the total discounted cost are related to the renewal function and the discounting function

Stochastic Degradation Models

- Total degradation caused by random shocks can be modelled as a compound process

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

- The distribution of $Y(t)$ in a general setting is not possible to derive

Stochastic Gamma Process

- It is a limiting form of a compound Poisson process
  - Increments are independent and gamma distributed
  - Cumulative degradation is also gamma distributed
- For these reasons, it is a widely applicable model

Moments of the Damage Cost

- Explicit expressions for a renewal process model

Table: Total Discounted Cost

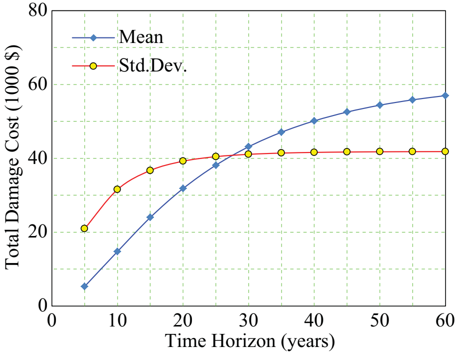
Moment	Expression
Mean	$\mathbb{E}\left[K_D(t)\right] = \mu_C \int_0^t e^{-\rho x}\, \mathrm{d}\Lambda(x)$
Mean square	$\mathbb{E}\left[K_D^2(t)\right] = \mu_{2C} \int_0^t e^{-2\rho x}\, \mathrm{d}\Lambda(x) + 2\mu_C \int_0^t e^{-2\rho x}\, \mathbb{E}\left[K_D(t-x)\right]\, \mathrm{d}\Lambda(x)$

Table: Total Cost

Mean	$\mathbb{E}\left[K(t)\right] = \mu_C \Lambda(t)$
Mean square	$\mathbb{E}\left[K^2(t)\right] = \mu_{2C}\Lambda(t) + 2(\mu_C)^2 \int_0^t \Lambda(t-x)\, \mathrm{d}\Lambda(x)$

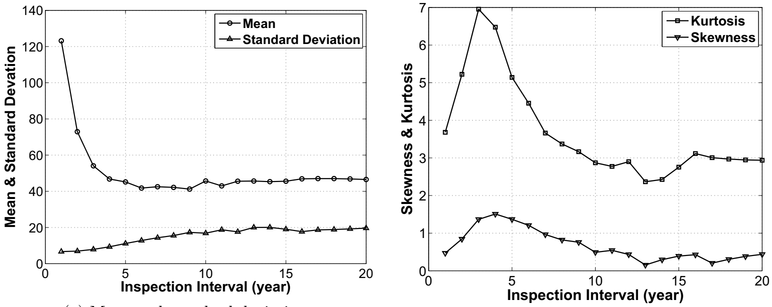
Notes: C and T are independent, $\Lambda(x)$ is the renewal function

Example 1: Total Damage Cost with Discounting



- Compound process model
  - A recurring hazard with gamma distributed inter-arrival time (mean: 25 years), and expected cost per event is 100 K$ (COV = 0.1), and discount rate as 5 % per year
  - The planning horizon is varied from 5 to 60 years

Example 2: Moments of Maintenance Cost

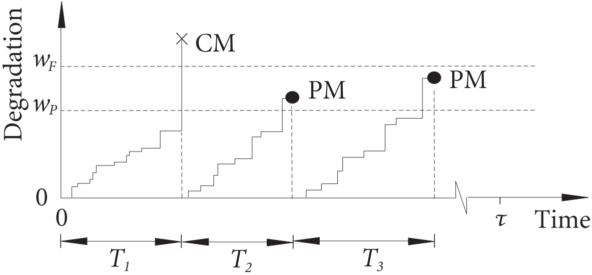


(a) Mean and standard deviation

(b) Skewness and kurtosis

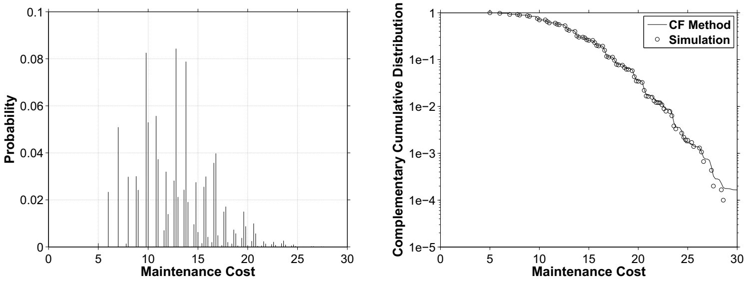
- Gamma process model of degradation
    - First 4 moments of the total maintenance cost versus the inspection interval
    - Costs of periodic inspection, preventive replacements and potential failures are included
    - Uncertain initiation of defects, followed by random growth

Example 3: Condition-Based Maintenance



- A renewal process driven by a gamma process
- A direct analytical solution is not possible
- The characteristic function is derived via an integral equation
- The characteristic function is inverted to obtain the probability distribution

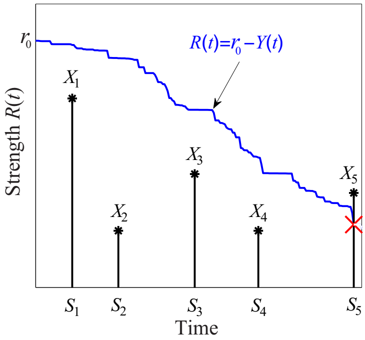
Distribution of the Total Maintenance Cost



(a) PMF (b) Complementary cumulative distribution

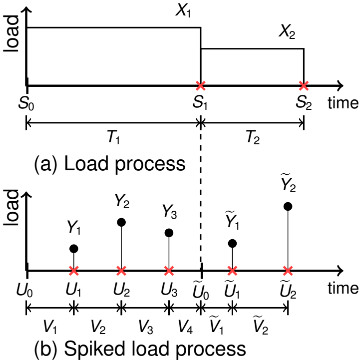
- Degradation as a gamma process, preventive maintenance policy, corrective maintenance upon a failure
- The total cost can be modelled as a discrete random variable
- The total cost as a multiple of the GCD of (c_I, c_P, c_F)

Stochastic Load and Degradation Combination



- A difficult class of problems
- The Poisson shock and gamma process can be combined
- Solution is based on a stochastic function of the gamma process

Load Combination Problem ongoing work



- Combination of a pulse process with a shock process
- Regeneration of the combined process at time S_1
- The distribution of maximum of the combined process is derived

Concluding Remarks

- An overview of the theory of renewal processes and related models is presented
- Applications to evaluate reliability and other measures of life-cycle performance are discussed
  - The regenerative property is a key to solving a variety of problems
  - Generalized solutions are presented for many problems which were analyzed using the Poisson process model
- Accurate solutions are derived for a finite time horizon with no restrictions on the type of distributions
- Applications to risk and reliability assessment of systems nearing the end of life are presented