Recall:

- Hamiltonian formulations are suitable for quantization.
- Lagrangian formulations are suitable to achieve general relativistic covariance.

**Strategy:**

1. SR, 1st Q. Lagrangian formulation
2. GR, 1st Q. Lagrangian formulation
3. GR, 2nd Q. Hamiltonian formulation

We already started step 1:

\[
H[\phi, \pi, t] = \frac{\delta H}{\delta \pi(x, t)} (T) \quad \text{and} \quad \pi(x, t) = \frac{\delta L}{\delta \dot{\phi}(x, t)} (T^{-1})
\]

Proposition: These equations of motion are equivalent:

- Hamiltonian eqns. of motion:
  \[
  \dot{\phi}(x, t) = \frac{\delta H[\phi, \pi, t]}{\delta \pi(x, t)} \quad (H1)
  \]
  \[
  \dot{\pi}(x, t) = -\frac{\delta H[\phi, \pi, t]}{\delta \phi(x, t)} \quad (H2)
  \]

- Lagrangian eqns. of motion:
  \[
  \dot{\phi}(x, t) = \frac{\delta L}{\delta \phi(x, t)} \quad (L1)
  \]
  \[
  \frac{\delta L}{\delta \dot{\phi}(x, t)} = \frac{d}{dt} \frac{\delta L}{\delta \phi(x, t)} \quad (L2)
  \]
Proof: We need to show that \((H1 \land H2) \iff (L1 \land L2)\).

The case "\(\implies\)"

**Show L1:** Indeed: \(\phi \overset{(H1)}{\implies} \frac{\delta \mathcal{L}}{\delta \pi} \overset{(T)}{\implies} \phi \checkmark\)

**Show L2:** Indeed:

\[
\frac{d}{dt} \frac{\delta \mathcal{L}(\phi, \pi, t)}{\delta \phi} \overset{(T^{-1})}{\implies} \frac{d}{dt} \pi
\]

\[
(\ref{H2}) - \frac{\delta \mathcal{H}(\phi, \pi, t)}{\delta \phi}
\]

\[
\overset{\text{by dy.}}{\implies} - \frac{\delta}{\delta \phi} \left( \int \beta(x, \pi) \rho(x) dx - L(\phi, \pi; \phi, \pi), t \right)
\]

\[
= \frac{\delta \mathcal{L}}{\delta \phi} - \frac{\delta \mathcal{K}}{\delta \pi} + \frac{\delta \mathcal{L}}{\delta \pi} \overset{\text{by dx.}}{\implies} \phi \checkmark
\]

The case "\(\iff\)" Exercise.

Result so far:

- Legendre transform to Lagrangian formulation

\[\implies\] Equations of motion can be cast in the form \(L1, L2\), i.e.:

\[
\left(\text{Note: Only a time derivative, no occurrence of space derivatives}\right) \rightarrow \frac{\delta L}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}(x,t)} , \quad \beta(x,t) = \dot{\phi}(x,t)
\]

But: How is that advantageous? These equations still seem to treat time differently than space!
Analysis of $L_1, L_2$:

We notice: $\star$ The term $\frac{\delta L}{\delta \phi(x,t)}$ is the total derivative with respect to all occurrences of $\phi$ in $L$, including occurrences of $\frac{\partial}{\partial x_i} \phi(x,t)$ in $L$.

Why? Because of the definition of $\frac{\delta}{\delta \phi}$:

$$\frac{\delta L}{\delta \phi(x,t)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ L[\delta \phi(x,t) + \varepsilon \delta \phi(x,t)] - L[\delta \phi(x,t)] \right]$$

E.g.: $F[u] := \int \sin(x) \left( \frac{d}{dx} u(x) \right) dx$ \text{ Is } \frac{\delta F}{\delta u(x)} = 0? \text{ No:} \frac{\delta F}{\delta u(x)} = -\cos(x) \Rightarrow \frac{\delta F}{u(x)} = -\cos(x)$

$\Rightarrow L_1, L_2$ will contain nontrivial time and space derivatives.

$\star$ Is there a systematic way to evaluate the derivatives with respect to $\frac{\partial}{\partial x_i}$?

Lemma: Consider any functional $Z$ of the form:

$$Z[f] = \int \text{ polynomial } \left( \frac{d}{dx} f \right) dx$$

Then: $\frac{\delta Z}{\delta f(x)} = -\frac{d}{dx} \frac{\delta Z}{\delta \left( \frac{d}{dx} f \right)}$ On the right hand side we view $\frac{d}{dx} f$ as an independent function.
Example:

Notation: $\frac{\partial f(x)}{\partial x} = \frac{d}{dx} f(x)$

$$Z[f] := \int_{\mathbb{R}} (\frac{\partial f(x)}{\partial x})^2 dx'$$

- If we view $\frac{\partial f}{\partial x}$ as an independent function, then we obtain of course:

$$\frac{\delta Z[f]}{\delta f(x)} = 2 \frac{\partial f(x)}{\partial x}$$

- Our lemma claims, therefore:

$$\frac{\delta^2 Z[f]}{\delta f(x)} = -2 \frac{\delta^2}{\delta (\frac{\partial f(x)}{\partial x})} = -2 \frac{\partial^2 f(x)}{\partial x^2}$$

Let us verify this from first principles!

Indeed:

$$\frac{\delta}{\delta f(x)} Z[f] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{\mathbb{R}} \left( (\frac{\partial f(x)}{\partial x} + \epsilon \delta(x-x')) \right)^2 dx' - \int_{\mathbb{R}} (\frac{\partial f(x)}{\partial x})^2 dx' \right]$$

$$\lim_{\epsilon \to 0} = 2 \int_{\mathbb{R}} (\frac{\partial^2 f(x)}{\partial x^2}) \delta(x-x') dx'$$

with L'Hôpital's rule:

$$= -2 \int_{\mathbb{R}} (\frac{\partial^2 f(x)}{\partial x^2}) \delta(x-x') dx' + \text{boundary form}$$

$$= -2 \frac{\partial^2 f(x)}{\partial x^2}$$
Recall L2: \[
\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{d}{dt} \frac{\delta L[\phi, \beta, t]}{\delta \beta(x,t)}
\]

Use lemma:

\[
\frac{\delta L[\phi, \beta, t]}{\delta \phi(x,t)} = \frac{\delta L[\phi, \beta, \phi, \phi, \phi, \beta, \beta]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{2}{\delta x^j} \frac{\delta L[\phi, \beta, \phi, \phi, \phi, \beta, \beta]}{\delta (\dot{\beta} x^j(x,t))}
\]

\[\Rightarrow L2 \text{ takes the form:}\]

\[
\frac{\delta L[\phi, \beta, \phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{2}{\delta x^j} \frac{\delta L[\phi, \beta, \phi, \beta, \beta]}{\delta (\dot{\beta} x^j(x,t))} = \frac{d}{dt} \frac{\delta L[\phi, \beta, \phi, \beta, \beta]}{\delta \beta(x,t)}
\]

Recall also L1: \[\beta(x,t) = \dot{\phi}(x,t)\]

\[\Rightarrow \text{One is tempted to write:}\]

\[
\frac{\delta L[\phi, \beta, \phi, \beta, t]}{\delta \phi(x,t)} - \sum_{j=1}^{3} \frac{2}{\delta x^j} \frac{\delta L[\phi, \beta, \phi, \beta, \beta]}{\delta (\dot{\beta} x^j(x,t))} = \frac{d}{dt} \frac{\delta L[\phi, \beta, \phi, \beta, \beta]}{\delta \beta(x,t)} \quad \text{with: } \dot{\beta} = \frac{d}{dt} \]

However: Here, we must remember that here the true variable \[\dot{\beta}\], and that we can set \[\dot{\beta} = \dot{\phi}\] only after functional differentiation.
Ramification? Can we use the lemma to write
\[ \frac{\delta L}{\delta \phi(x,t)} = 0 \]
for the Euler-Lagrange field equations? No!

Because: to apply the lemma to the derivative \( \frac{\partial}{\partial x} \phi \), one would need that \( L \) possesses a \( t \)-integration:

\[ \text{(Lemma: For any functional } Z \text{ of the form:} \]
\[ Z[f] = \int \text{polynomial } \left( \frac{\partial f}{\partial x} \right) dx \]
\[ \text{we have:} \quad \frac{\delta Z}{\delta f(x)} = -\frac{dx}{\delta x} \frac{\delta Z}{\delta \left( \frac{\partial f}{\partial x} \right)} \]

\[ \Rightarrow \text{The "Action functional"} : \]

Definition: \[ S[\phi] := \int_L[\phi, t] dt \]

\( S[\phi] \) is called the "action of the field evolution \( \phi(x,t) \)"

Then, the "Euler-Lagrange field equations" are

\[ \sum_{n=3}^{2} \frac{\partial}{\partial x^n} \frac{\delta S[\phi_{x^n}]}{\delta (\partial_x \phi)} = 0 \]

or equivalently:

\[ \frac{\delta S[\phi]}{\delta \phi(x,t)} = 0 \]

"The action principle"
Notice that the action principle, spelled out, reads:

\[
O = \frac{\delta S[\phi]}{\delta \phi(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( S\left[\phi(x') + \varepsilon \delta(x-x_0)\right] \right)
- S\left[\phi(x')\right] \right)
\]

Example:

The Klein Gordon action:

\[
S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} (\partial_0 \phi)^2 - \sum_{j=1}^3 (\partial_j \phi)^2 - m^2 \phi^2 \, d^4x
\]

Using either the action principle or directly the Euler-Lagrange field equations, one obtains indeed the Klein Gordon equation (Exercise: verify):

\[
\partial_0^2 \phi - \Delta \phi + m^2 \phi = 0 \quad \text{i.e.} \quad (\square + m^2) \phi(x,\varepsilon) = 0
\]

Definitions:

* The action's integrand is called the "Lagrange density" \(L(x,\varepsilon)\):

\[
S'[\phi] = \int_{\mathbb{R}^4} L(x,\varepsilon) \, d^4x
\]

* One often formally writes:

\[
\frac{\partial L}{\partial \phi} - \sum_{\mu=0}^3 \partial_\mu L = 0 \quad \text{(L)}
\]
* Notation often used in General Relativity:

1. \( \phi_{\mu}(x,t) := \frac{2}{\partial x^\mu} \phi(x,t) \)

2. Twice occuring indices are to be summed over (Einstein summation convention):

   E.g., equation (1) can be written as:

   \[
   \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial x^\mu \phi} = 0
   \]

3. One defines the metric tensor \( g_{\mu\nu}(x,t) \).

   More about it soon. In special relativity in inertial rectangular coordinate system, we have:

   \[
   g_{\mu\nu}(x,t) = \eta_{\mu\nu} = \begin{pmatrix}
   1 & 0 & 0 & 0 \\
   0 & -1 & 0 & 0 \\
   0 & 0 & -1 & 0 \\
   0 & 0 & 0 & -1
   \end{pmatrix}
   \]

   □ Using these definitions, the K.C. action now reads:

   \[
   S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{\mu} \phi_{\nu} - m^2 \phi^2 \, d^4x
   \]

   The inverse metric to \( g_{\mu\nu} \). In special relativity, both are the same \( (\eta_{\mu\nu}) \)

   □ The E.L. eqns read

   \[
   \frac{\delta S[\phi(x,t)]}{\delta \phi(x,t)} = \partial_{\mu} \frac{\delta S[\phi(x,t)]}{\delta (\phi_{\mu}(x,t))}
   \]

   and yield \( -m^2 \phi = \partial_{\mu} g^{\mu\nu} \phi_{\nu} \)

   i.e., of course: \( (\Box + m^2) \phi = 0 \)
We have now completed Step 1:

**SR, 1st Q.**
Hamiltonian formalism

→
Legendre transform (equivalence)

→
**SR, 1st Q.**
Lagrangian formalism

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**GR, 1st Q.**
Hamiltonian formalism

→
Legendre transform (equivalence)

→
**GR, 1st Q.**
Lagrangian formalism

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**GR, 2nd Q.**
Hamiltonian formalism

←
Ossian Schmidt: Covariant geometrical

→
**GR, 2nd Q.**
Lagrangian formalism


Now that we have a beautifully covariant Lagrangian formulation:

Step 2: How to allow for curvature of space-time?

**Strategy:**

A. Within special relativity, allow not just inertial rectangular coordinate systems but allow arbitrary coordinate systems.

B. Allow arbitrary coordinate systems and allow curvature.
A. Arbitrary coordinate systems

1. Reconsider the L.C. action:

\[ S[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \, \mathrm{d}^4 x \]

2. If we change to arbitrary coordinates

\[ x^\mu \rightarrow \tilde{x}^\nu = \tilde{\phi}(x) \]

then:

\[ \phi(x) \rightarrow \tilde{\phi}(\tilde{x}) = \phi(x(\tilde{x})) \quad (\text{recall that } \tilde{x}_0 \text{ is simplified}) \]

\[ \frac{\partial}{\partial x^\mu} \phi(x) \rightarrow \frac{\partial}{\partial \tilde{x}^\nu} \tilde{\phi}(\tilde{x}) = \left( \frac{\partial}{\partial x^\nu} \phi(x(\tilde{x})) \right) \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \]

3. Therefore, if we transform

\[ g^{\mu\nu}(x) \rightarrow \tilde{g}^{\nu\sigma}(\tilde{x}) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} g^{\rho\phi}(x(\tilde{x})) \]

then we have that this term in the action

\[ g^{\mu\nu}(x) \phi_{,\mu}(x) \phi_{,\nu}(x) \]

is numerically the same in all coordinate systems:

\[ g^{\mu\nu}(x) \left( \frac{\partial}{\partial x^\mu} \phi(x) \right) \left( \frac{\partial}{\partial x^\nu} \phi(x) \right) \rightarrow \tilde{g}^{\nu\sigma}(\tilde{x}) \left( \frac{\partial}{\partial \tilde{x}^\nu} \tilde{\phi}(\tilde{x}) \right) \left( \frac{\partial}{\partial \tilde{x}^\sigma} \tilde{\phi}(\tilde{x}) \right) \]

\[ = \tilde{g}^{\nu\phi}(\tilde{x}) \left( \frac{\partial}{\partial \tilde{x}^\nu} \tilde{\phi}(\tilde{x}) \right) \left( \frac{\partial}{\partial \tilde{x}^\phi} \tilde{\phi}(\tilde{x}) \right) \]

\[ = g^{\mu\nu}(x) \left( \frac{\partial}{\partial x^\mu} \phi(x) \right) \left( \frac{\partial}{\partial x^\nu} \phi(x) \right) \quad \text{because} \quad \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial x^\phi}{\partial \tilde{x}^\nu} = \delta^\nu_\phi \]
**Terminology:**

* We say that we let $g^{\mu\nu}(x)$ transform as a contravariant tensor of rank 2.

* With $g^{\mu\nu}(x) g_{\nu\rho}(x) = \delta^\mu_\rho$ we have

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^d}{\partial \tilde{x}^\mu} \frac{\partial x^b}{\partial \tilde{x}^\nu} g_{\mu\nu}(x(\tilde{x}))$$

which is called a covariant rank 2 tensor.

**Is $S[\phi]$ now coordinate system independent?**

No, not yet!

**Recall:**

As $x^\mu \rightarrow \tilde{x}^\mu(x)$ the integral measure

changes by a Jacobian factor:

$$\int f(x) \, d^nx \rightarrow \int \tilde{f}(\tilde{x}) \, det \left( \frac{\partial x^b}{\partial \tilde{x}^\mu} \right) \, d^nx$$

\[\tilde{f}(\tilde{x}) \text{ a coordinate-dependent term!}\]

**A compensating term is needed:**

How can we modify the action $S[\phi]$ so that:

* there is no modification in cartesian coordinates

* the modification compensates the Jacobian term.
\textbf{Solution:}

Modify the action to include a "Volume factor":

\[ S[\phi] := \frac{1}{2} \int_{\mathbb{R}^4} \left( g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right) \sqrt{-\det(g_{\mu \nu})} \, d^4x \]

\textbf{The volume factor:}

\begin{itemize}
  \item When \( g_{\mu \nu} = \begin{pmatrix} 1 & \gamma \nabla \gamma \\ 0 & -1 \end{pmatrix} \) then \( \sqrt{-\det(g_{\mu \nu})} = 1 \) \( \checkmark \)
  \item Lemma: When \( x^\mu \rightarrow \tilde{x}^\mu(x) \) then:
    \[ \sqrt{-\det(g_{\mu \nu})} \rightarrow \sqrt{-\det(\tilde{g}_{\mu \nu})} = \det \left( \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right) \sqrt{-\det(g_{\mu \nu})} \]
\end{itemize}

\textbf{Therefore, we have now in special relativity that the action \( S[\phi] \) of a field \( \phi \) comes out the same number, independently of one's choice of coordinate system:}

\[ S[\phi] \rightarrow \tilde{S}[\tilde{\phi}] = \int \sqrt{-\tilde{g}} \det \left( \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right) \sqrt{-\det(g_{\mu \nu})} \, d^4x \]

\[ = \int \det \left( \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right) \det \left( \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right) \sqrt{-\det(g_{\mu \nu})} \, d^4x \]

\[ = \int \det(\delta^\mu_\nu) \sqrt{-\det(g_{\mu \nu})} \, d^4x = \int \sqrt{-\det(g_{\mu \nu})} \, d^4x = S[\phi] \]
B. How to allow curvature?

* The trivial metric $g_{\mu\nu}(x) = g_{\mu\nu} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$

  can look very nontrivial in generic coordinate systems: $g_{\mu\nu}(x) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$

* But: Some metrics $g_{\mu\nu}(x)$ are not obtainable from the trivial metric by a coordinate change!

  These metrics belong to spaces with curvature. We need not change the action's formula: just allow arbitrary metrics $g_{\mu\nu}(x)$. 