Note Title

Recall the strategy:

SR, 1^{st}Q Hamiltonian formalism	$\xrightarrow[\text{(equivalence)}]{\text{step 1: Legendre transform}}$	SR, 1^{st}Q. Lagrangian formalism	we are here
		$\downarrow$ step 2: allow curvature	
GR, 1^{st}Q Hamiltonian formalism	$\xleftarrow[\text{(equivalence)}]{\text{step 3: Legendre transform}}$	GR, 1^{st}Q Lagrangian formalism	
$\downarrow$ step 4		$\downarrow$	
GR, 2^{nd}Q Hamiltonian formalism	$\xleftrightarrow[\text{(equivalence)}]{\text{Dyson Schwinger eqns are same}}$	GR, 2^{nd}Q Lagrangian formalism (Path integral of QFT)	

Step 2 so far:

- We started with the Klein Gordon action in special relativity:

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\,\phi_{,\nu} - m^2\phi^2 - V(\phi,\varphi_i)\;d^4x \qquad (a)$$

This formulation is correct for any inertial observer using a rectangular coordinate system, and only for those observers.

Remark: Here, $V(\phi,\varphi_i)$ is a potential. It describes how the ϕ field interacts with other fields $\{\varphi_i\}$ and with itself. The very early universe is assumed to have been dominated by a scalar field ϕ which interacted mainly with itself, i.e., one had effectively: $V(\phi) = \lambda_1\phi^3 + \lambda_2\phi^4 + \dots$

We then developed a formulation of the Klein Gordon action in special relativity which is the same in all coordinate systems:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4} \left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi, \ell_i)\right) \overbrace{\sqrt{|g|}}^{\sqrt{-\det(g_{\mu\nu})}}\, d^4x \qquad (b)$$

- If an inertial observer sets up a rectangular coordinate system, then the metric tensor $g_{\mu\nu}(x)$ is of course given by this constant matrix

$$g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

with $\sqrt{|g|} = 1$. Thus, one recovers equation (a).

- But we saw that in generic (i.e. arbitrarily chosen) coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the metric tensor $\tilde{g}_{\mu\nu}(\tilde{x})$ is given by:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta} \qquad (c)$$

$\Rightarrow$ In special relativity, in arbitrary coordinates, the metric $g_{\mu\nu}$ is a position-dependent matrix of the form (c).

* We notice that $g_{\mu\nu}(x)$ is always symmetric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

Key Question:

Can any arbitrary function obeying $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ arise from

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

by changing coordinates according to $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} \eta_{\alpha\beta}$?

Answer: No ! The others describe "curved" spacetimes.

A given spacetime can be described by any one of an equivalence class $[g]$ of metric functions $\{g_{\mu\nu}(x)\}$, which differ by a mere change of coordinates (i.e. which are related by a diffeomorphism).

Definition: Each equivalence class $[g]$ is called a Riemannian or Lorentzian Structure, depending on the signature of the metric.

How many Lorentzian or Riemannian structures are there?

Q: How many independent degrees of freedom D (i.e. independent functions) describe a spacetime fully?

A: In n dimensions, the metric g has n^2 component functions $g_{\mu\nu}(x)$.

Because of $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, only $n(n+1)/2$ are independent.

But we can choose n functions $\tilde{x}^\mu(x)$ in $\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha(\tilde{x})}{\partial \tilde{x}^\mu} \frac{\partial x^\beta(\tilde{x})}{\partial \tilde{x}^\nu} g_{\alpha\beta}$.

$$\Rightarrow \quad \boxed{D = n(n+1)/2 - n}$$

Examples: For $n = 1+3$, have $D = 6$.

For $n = 2$, have $D = 1$.

Curvature:

- We continue to postulate the coordinate system-independent Klein Gordon action of above:

$$S_{KG}[\phi] := \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - V(\phi,\varphi_i)\right)\sqrt{|g|}\, d^4x$$

- We will allow almost arbitrary metric tensors $g_{\mu\nu}(x)$, even those for which there do not exist coordinates $\tilde{x}$ in which:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x$$

- But we must have that, at least locally, special relativity holds!

$\Longrightarrow$ Consider only $g_{\mu\nu}(x)$ for which for each x_0, there exists a change of coordinates

$$x \to \tilde{x}$$

so that: $$\tilde{g}_{\mu\nu}(\tilde{x}_0) = \eta_{\mu\nu}$$

- This requirement is **The Equivalence Principle**:

  * We postulate that gravity can always locally be eliminated:

  * We assume that if a freely falling observer in a small region sets up a rectangular coordinate system the observer will see arbitrarily small gravity effects if the region is made arbitrarily small.

  * For this to be true, any body's gravitational mass must be equal to its inertial mass, i.e. all bodies must fall equally. (Else the notion of freely falling observer) is not even well defined)

How can one identify the presence of curvature?

* Assume we are given a metric tensor

$$g_{\mu\nu}(x)$$

as an explicit matrix-valued function, in some coordinates.

* How can we determine whether or not this is, e.g., the metric tensor of flat space-time, i.e., whether or not there exist coordinates $\tilde{x}$ in which $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$?

* This problem is solved in differential geometry:

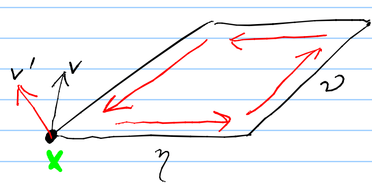
Define: The "Christoffel symbol functions":

$$\Gamma^{\mu}_{\alpha\beta}(x) := \frac{1}{2} g^{\mu\nu}(x)\left(g_{\alpha\nu,\beta}(x) + g_{\beta\nu,\alpha}(x) - g_{\alpha\beta,\nu}(x)\right)$$

Define: The "Riemann Curvature Tensor":

$$R^{i}{}_{jkl}(x) := \Gamma^{i}_{lj,k}(x) - \Gamma^{i}_{kj,l}(x) + \Gamma^{s}_{lj}(x)\Gamma^{i}_{ks}(x) - \Gamma^{s}_{kj}(x)\Gamma^{i}_{ls}(x)$$

It's rôle? A space is called curved at x if the parallel transport of a vector v along an infinitesimal parallelogram returns the vector v' to x, but v' is rotated by some amount. $R^{\mu}{}_{\nu\alpha\beta}$ tells by how much:



$$\boxed{(v'-v)^a = \eta^b \nu^c R^a{}_{bcd}(x)\, v^d}$$

Remark:

If the parallelogram does not even close we say that space-time has "Torsion". There is no evidence for torsion in nature.

Proposition:

Assume that, in a region, A, of space-time:

$$R^{\mu}{}_{\nu\rho\varepsilon}(x) = 0 \quad \text{for all } x \in A$$

Then and only then there exist coordinates $\tilde{x}$ so that:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu} \quad \text{for all } x \in A$$

The dynamics of space-time

Problem: What are the equations of motion for spacetime's curvature?

Which are the degrees of freedom of curvature for which we have to find an equation of motion?

- We saw that the curvature of space-time is encoded in the matrix-valued metric function:

$$g_{\mu\nu}(x)$$

- However, if $g_{\mu\nu}(x)$ looks nontrivial, this can be for two different reasons:

1. Spacetime has little or no curvature and $g_{\mu\nu}(x)$ is nontrivial just because of an unlucky choice of coordinates.

2. Spacetime is curved, i.e., we can not make $g_{\mu\nu}(x)$ take the form $\tilde{g}_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$ for all $\tilde{x}$ no matter which coordinates we choose.

- Therefore, it is difficult to pinpoint in the matrix function $g_{\mu\nu}(x)$ the curvature degrees of freedom.
- And: Even the entries $R^{\mu}{}_{\nu\delta\varepsilon}$ of the curvature tensor are coordinate system dependent.

Strategy:

- Use the degrees of freedom of curvature to build a scalar and therefore coordinate system independent function $S_{g_{\mu\nu}}[g]$
- Then, use this function as the action for gravity.
- The equations of motion for gravity should follow from the action principle (and they do).

$\rightsquigarrow$ Need to define a scalar that encodes curvature!
We begin by going from a 4-tensor to a 2-tensor:

Definition: The "Ricci tensor":

$$\boxed{R_{\mu\nu}(x) := R^{i}{}_{\mu i \nu}(x)}$$

Recall: $\sum_{i=0}^{3}$ is implied

Note: Other index contractions would vanish because of antisymmetries of $R^{\mu}{}_{\nu\rho\varepsilon}(x)$ that are implied by the definition of $R^{\mu}{}_{\nu\rho\varepsilon}(x)$.

Remark:

$R_{\mu\nu}(x)$ carries strictly less information than the full Riemann curvature tensor:

* If $R_{\mu\nu}(x)=0$ it is still possible that $R^{\mu}{}_{\nu\rho\varepsilon}(x)\neq 0$!
* This happens to be the case, e.g., for gravitational waves.

Definition: The "curvature scalar" (or "Ricci scalar")

$$R(x) := g^{\mu\nu}(x)\, R_{\mu\nu}(x)$$

Other curvature scalars:

* The simplest scalar that can be formed from the metric alone is $g^{\mu\nu}(x)\, g_{\nu\mu}(x) = 4$.
* The next simplest scalar that can be formed is the Ricci scalar $R(x)$.
* All other scalars made out of g only are composed of higher powers of the Riemann tensor $R^{\mu}{}_{\nu\rho\varepsilon}(x)$:

$$R_{\mu\nu}R^{\mu\nu}, \quad R^{\cdot\cdot}_{\cdot\cdot}R^{\cdot\cdot}_{\cdot\cdot}R^{\cdot\cdot}_{\cdot\cdot} \quad \text{etc.}$$

The gravitational action

- A priori, the full action now reads:

other "matter" fields for e^-, quarks, photons etc

$$S_{tot}[g, \phi, \varphi_i] = S_{KG} + S_{other} + S_{grav}$$

Klein Gordon action

with: $S_{grav}[g] := \int \left(c_0 + c_1 R(x) + c_2 R_{\mu\nu}(x) R^{\mu\nu}(x) + c_3 R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot}_{\cdot\cdot} R^{\cdot\cdot} + \ldots \right) \sqrt{|g|}\, d^4x$

- Comparison with experiment shows evidence only for the first two terms:

$$S_{grav}[g] = -\frac{1}{16\pi G} \int (2\Lambda + R(x)) \sqrt{|g(x)|}\, d^4x$$

"Einstein action"   Newton's constant   "Cosmological constant"

Remark: D. Lovelock here determined all generalizations to higher terms and higher dimensions that still possess 2^{nd} order initial value problems.

at Applied Math at UW

The equations of motion

The action principle is to require that the action be extremal with respect to all degrees of freedom:

A) $\frac{\delta S_{tot}}{\delta \varphi_i(x)} = 0$ B) $\frac{\delta S_{tot}}{\delta g_{\mu\nu}(x)} = 0$ C) $\frac{\delta S_{tot}}{\delta \phi(x)} = 0$

A) **Require:** $\frac{\delta S_{tot}[g, \varphi, \phi]}{\delta \varphi_i(x)} = 0$

This yields the general relativistically covariant field equations for all "other" fields. (We will ignore the $\varphi_i(x)$ for now.)

Quantization: Legendre transform $\rightarrow H(\varphi_i, \pi_{\varphi_i}) \longrightarrow$ impose $[\varphi_i, \pi_{\varphi_j}] = i\delta$ etc.

B) Require: $\frac{\delta}{\delta g_{\mu\nu}(x)} S_{tot}[\phi, e_i, g] = 0$

This yields the equation of motion for the dynamics of curvature, i.e., the Einstein equation:

(See exercise in Mukhanov's text) $\rightarrow$

$$\boxed{\underbrace{R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x)}_{\sim \frac{\delta S'_{grav}}{\delta g_{\mu\nu}(x)}} = \underbrace{+ 8\pi G\, T_{\mu\nu}(x)}_{\sim - \frac{\delta(S_{other} + S'_{KG})}{\delta g_{\mu\nu}(x)}}}$$

* Here, $T_{\mu\nu}(x)$ is the "Energy Momentum Tensor". Neglecting the contribution by the $e_i(x)$, one obtains:

$$T^{(K.G.)}_{\mu\nu}(x) = \phi_{,\mu}(x)\phi_{,\nu}(x) - g_{\mu\nu}(x)\Big(\frac{1}{2} g^{\alpha\beta}(x)\,\phi_{,\alpha}(x)\,\phi_{,\beta}(x) - \overbrace{V(\phi(x))}^{\text{mass term } m^2\phi^2 \text{ included}}\Big)$$

* **Quantization:** To quantize the Einstein equation is difficult for many reasons:

- For example, it is difficult to separate the curvature degrees of freedom from mere artifacts of the choice of the coordinate system.
- Also, the Einstein equation is highly nonlinear.
- So far, all attempts have run into severe difficulties, even perturbative approaches.

$\rightsquigarrow$ **This course:** 1.) We will here initially consider known classical solutions $g_{\mu\nu}(x)$ and quantize only $\phi(x)$.

2.) Then, we will quantize linear perturbations of the metric.

c) Require: $\frac{\delta}{\delta\phi(x)} S_{tot}[g,\phi] = 0$

- Since ϕ occurs only in S_{KG} we have, equivalently:

$$\frac{\delta S_{KG}}{\delta \phi(x)} = 0$$

- Recall S_{KG}:

$$S_{KG}[\phi] = \frac{1}{2}\int_{\mathbb{R}^4}\Big(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \underbrace{\lambda\phi^4}_{\text{Example of a potential}}\Big)\sqrt{|g|}\, d^4x$$

- Apply the Euler Lagrange equations:

$$\frac{\delta S_{KG}[\phi,\{\phi_{,\mu}\},g]}{\delta\phi(x,t)} = \partial_\mu \frac{\delta S[\phi,\{\phi_{,\mu}\},g]}{\delta(\phi_{,\mu}(x,t))}$$

- $\Rightarrow$ Klein Gordon equation in general relativity:

$$\left(-\frac{1}{2}m^2 2\phi(x) - \frac{1}{2}\lambda 4\phi^3(x)\right)\sqrt{|g(x)|} = \partial_\mu\left(\frac{2}{2}g^{\mu\nu}(x)\,\phi_{,\nu}(x)\sqrt{|g(x)|}\right)$$

Thus:

$$\boxed{\frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(x)\sqrt{|g(x)|}\,\phi_{,\nu}(x)\right) + m^2\phi(x) + 2\lambda\phi^3(x) = 0}$$

- Definition: The "d'Alembert operator", $\Box$

$$\Box := \frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^\mu}\, g^{\mu\nu}(x)\sqrt{|g(x)|}\frac{\partial}{\partial x^\nu}$$

Thus:

$$\boxed{\Box\,\phi(x) + m^2\phi(x) + 2\lambda\phi^3(x) = 0}$$

Next: Step 3 in

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{SR, } 1^{st}\text{Q Hamiltonian formalism} \arrow[r, "\text{step 1: Legendre transform}", "\text{(equivalence)}"'] & \text{SR, } 1^{st}\text{Q. Lagrangian formalism} \arrow[d, "\text{allow curvature}", "\text{step 2}"'] \\
\text{GR, } 1^{st}\text{Q Hamiltonian formalism} \arrow[d, "\text{step 4}"] & \text{GR, } 1^{st}\text{Q Lagrangian formalism} \arrow[l, "\text{step 3: Legendre transform}"', "\text{(equivalence)}"] \arrow[d, dashed, "\text{(PI)}"] \\
\text{GR, } 2^{nd}\text{Q Hamiltonian formalism} \arrow[r, leftrightarrow, "\text{Dyson Schwinger eqns are same}", "\text{(equivalence)}"'] & \text{GR, } 2^{nd}\text{Q Lagrangian formalism (Path integral of QFT)}
\end{tikzcd}
$$

Comment on step (PI): 2^{nd} quantization with path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system.
- Then, for each field $\phi(\vec{x},t)$ we can calculate its action $S_{KG}[\phi,g]$:

$$S'_{KG}[\phi,g] = \frac{1}{2}\int_{\mathbb{R}^4}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2 - \lambda\phi^4\right)\sqrt{|g|}\, d^4x$$

- Following Feynman, we obtain "probability amplitudes":

$$\text{prob. ampl.}[\phi] := e^{\frac{i}{\hbar}S'_{KG}[\phi,g]}$$

- Consider, e.g., the vacuum expectation value of $\phi(\vec{x},t)\,\phi(\vec{x}',t)$, i.e., the correlation function of field amplitudes:

$$G(\vec{x},t,\vec{x}',t) := \langle 0|\, \hat{\phi}(\vec{x},t)\, \hat{\phi}(\vec{x}',t)\, |0\rangle$$

- We will later see how to calculate it using commutation relations etc.

- With Feynman we also get it from the path integral:

Advantages:

1) Quick derivation of Feynman rules

2) Manifestly covariant.

$$G(\vec{x},t,\vec{x}',t) = N \int_{\text{all }\phi} \phi(\vec{x},t)\,\phi(\vec{x}',t)\; e^{\frac{i}{\hbar} S_{KG}[\phi,g]}\, D[\phi]$$

↑ "Path Integral" over a function space

Problems:

1) Ill defined due to uncountable number of integrations and related divergencies.

2) Even when these are temporarily regularized, the identification of the vacuum state is ambiguous.

→ This issue is better handled in canonical formalism.