Quantum field theory on FRW spacetimes.

**Observations:**

On scales $\gg 1$ Gyr:

- The universe is spatially very flat.
- The cosmic expansion is very isotropic.

**Friedmann-Robertson-Walker (FRW) spacetimes:**

- Simplifying approximation:
  
  Spacetime is modeled as having
  - no spatial curvature at all.
  - entirely isotropic expansion.

**Remark:**

It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Weinberg & Ellis.

There are even solutions that only temporarily get very close to flatness. The Einstein equs are nonlinear!
With these assumptions, we choose convenient coordinates:

**Time coordinate** $t$: 

Definition: The motion of galaxies due to the cosmic expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

Definition: As the time coordinate, $t$, let us use the proper time, $\tau$, of a freely streaming observer who has no peculiar velocity.

(to a good approximation, you can use your worst watch on earth)

**Space coordinates:**

It is convenient to use "comoving coordinates", $x_1, x_2, x_3$:

- At one time, $t_0$, (say today) we set up an ordinary rectangular coordinate system.
- Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

**Advantages:**
- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.
- Waves keep their wavelengths numerically constant even while they get physically stretched.
The metric:

Recall that \( ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu \)

is the invariant 4-distance.

In our coordinates, \( g_{\mu\nu}(x) \) must read:

\[
g_{\mu\nu}(t, x) = \begin{pmatrix} 1 & -a'(t) \\ -a'(t) & -a^2(t) \end{pmatrix}
\]

because we use units which "proper" time

because our coordinate system's unit of length means over time

\( a \) larger and larger proper length.

The "scale factor":

- The scale factor function \( a(t) \) is needed to take into account the expansion when calculating distances.

- Example: The proper distance between two galaxies with comoving distance \( (\Delta x_1, \Delta x_2, \Delta x_3) \) at proper time \( t \) is:

\[
d = \sqrt{g_{\mu\nu}(x_0) \, dx^\mu \, dx^\nu} = a(t) \sqrt{\left(\Delta x_1\right)^2 + \left(\Delta x_2\right)^2 + \left(\Delta x_3\right)^2}
\]

Note: \( \Delta x_0 = t_2 - t_0 = 0 \) since we are looking at the distance between the galaxies at equal times.
**Dynamics of a (t):**

The function $a(t)$ is determined by all equations of motion:

1. Calculate the energy momentum tensor $T_{\mu\nu}(x,\overline{x})$

   contributions of at least the most important fields, say $\phi_i(x,\overline{x})$.

2. Solve, simultaneously:

   * The equations of motion for the fields $\phi_i$.

   * The Einstein equation for $g_{\mu\nu}$, while setting $g_{\mu\nu}(x,\overline{x}) = \left(\begin{array}{cc} g_{00}(x,\overline{x}) & g_{01}(x,\overline{x}) \\ -g_{01}(x,\overline{x}) & \gamma_{11}(x,\overline{x}) \end{array} \right)$:

   $$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G T_{\mu\nu}(x)$$

**Semi-classical approximation**

We can solve these classically, but not quantum mechanically:

Can quantize only $\phi_i$, not $g_{\mu\nu}$.

$\Rightarrow$ need to "make quantum $T_{\mu\nu}(x,\overline{x})$ classical" for Einstein eqns!

$\Rightarrow$ one uses: $\overline{T}_{\mu\nu}(x) = \langle \text{QG} T_{\mu\nu}(x,\overline{x}) \rangle$

Problem: Energy & momentum are naturally nonlocal because of uncertainty principle.

Remark: $a(t)$ is related to curvature between speed & time.

For now, we will assume that the expansion's scale factor function $a(t)$ is given.
Convenient Definition: The conformal time coordinate, $\eta$.

Recall that:

$$g_{\mu\nu}(t, x) = \begin{pmatrix} 1-a^2(t) & 0 \\ -a^2(t) & -a^2(t) \end{pmatrix}$$

It would be convenient if $g_{\mu\nu}$ were proportional to $g_{\mu\nu} = \begin{pmatrix} 1-a^2 & 0 \\ 0 & -1 \end{pmatrix}$.

This can be achieved by choosing a new time coordinate $\eta$, so that time also has a prefactor $a^2$, i.e., so that:

$$(\Delta t)^2 = a^2(t)(\Delta \eta)^2$$

To this end, we need: $a\, d\eta = dt$

i.e.: $\frac{d\eta}{dt} = \frac{1}{a}$

and therefore $\eta(t) = \int \frac{1}{a(t')} \, dt'$ yields arbitrary integration constant.

The variable $\eta$ is called the "conformal time".

Because it shows that the FRW spacetime is equivalent to Minkowski space up to time-dependent conformal, i.e., angle-preserving, i.e. scale factor only transformations.

Using conformal time and removing spatial coordinates the metric reads:

$$g_{\mu\nu}(\eta, x) = a^2(\eta) \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = a^2(\eta) \, g_{\mu\nu}$$
This also implies:

\[ g^{\mu\nu}(\eta, \vec{x}) = a^{-2}(\eta) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = a^{-2}(\eta) \eta^{\mu\nu} \]

Recall: \( g^{\mu\nu} g_{\nu\nu} = \delta^{\mu}_{\nu} \), i.e., \( g_{\nu\nu} \) and \( g^{\mu\nu} \) are inverse to another.

We easily obtain the integral measure needed for the action:

\[ V_{gi} = \sqrt{\det(g_{\mu\nu}(\eta, \vec{x}))} = a^4(\eta) \]

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**The Klein Gordon field in FRW spacetimes**

Neglecting a potential \( V(\phi) \) for now, we obtain the action of the "free K.G. field on the FRW background":

\[ S_{\text{nc}} = \int \left( \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) V_{gi} \, d^4x \]

\[ \equiv \int \left( \frac{1}{2} a^{-2}(\eta) \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 \, d\eta \, d^3x \]

Thus, from the general Euler Lagrange equation:

\( \left( \frac{1}{V_{gi}} \frac{2}{\partial x^\mu} g^{\mu\nu} V_{gi} \frac{\partial}{\partial x^\nu} + m^2 \right) \phi(x) = 0 \)
\[
\left( \frac{1}{a^2(\eta)} \frac{d}{d\eta} \frac{d}{d\eta} \frac{\partial^2}{\partial \eta^2} + m^2 \right) \phi(\eta) = 0
\]

\[
\left( \frac{1}{a^2(\eta)} \gamma^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} + \frac{1}{a^2(\eta)} 2 \partial_\eta \frac{\partial}{\partial x^\eta} + m^2 \right) \phi(x) = 0
\]

\[
\phi''(\eta, \vec{x}) + \frac{2 \partial'(\eta)}{a(\eta)} \phi'(\eta, \vec{x}) - \Delta \phi(\eta, \vec{x}) + a^2(\eta) m^2 \phi(\eta, \vec{x}) = 0
\]

This is the K.G. eqn. in FRW spacetimes!

**Problem:** the equation above has this general form:

\[
\phi'' + \nabla^2 \phi + \dot{\phi} = 0
\]

- a time-dependent term
- a term that also occurs in the usual harmonic oscillator. Notice that this is entirely new.

**Strategy:** Use a new, re-scaled, field variable \( X \):

We try to change from \( \phi(\eta, \vec{x}) \) to a new field variable, say \( X(\eta, \vec{x}) \), so that the equation of motion for \( X \) has no "friction"-type term.

This simple ansatz succeeds:

\[
X(\eta, \vec{x}) := a(\eta) \phi(\eta, \vec{x})
\]

Namely:

we have:

\[
\phi' = \frac{2}{a} \frac{1}{a} X = -\frac{a'}{a^2} X + \frac{1}{a} X'
\]

and:

\[
\phi''_{\eta} = \frac{2}{a^2} \frac{1}{a(\eta)} X(\eta, \vec{x}) = \frac{1}{a} X_i; \text{ for } i=1,2,3
\]
Using these, the action in terms of $x$ becomes:

$$S_x = \int \frac{1}{2} \left( x'^2 - \sum \frac{3}{2} x_i^2 - \left( \frac{m^2 a^2 - a''}{a} \right) x^2 \right) dx d^3 x$$

Note that this term is the time-dependent mass term $m(x)$.  

Exercise: verify

**Equation of motion:**

* Do

$$\frac{\delta S}{\delta \phi(y^2)} = 0 \quad \text{and} \quad \frac{\delta S}{\delta x(y^2)} = 0$$

yield equivalent equations of motion?

* Yes, because:

$$0 = \frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta x} \frac{\delta x}{\delta \phi}$$

if $\delta S/\delta \phi$ vanishes then also $\delta S/\delta x$ vanishes.

* Thus, we may calculate the equation of motion directly in terms of $x$ from $S[x]$, to obtain:

Exercise: verify!  

$$x'' - \Delta x + \left( \frac{m^2 a^2 - a''}{a} \right) x = 0 \quad (EoM1)$$

**Remark:**  
We could have obtained this equation of motion directly from that of $\phi$ by change of variable.  But finding the action for $x$ was still worthwhile, namely to get the conjugate to $x$.  

A Preparation for quantization:

* We need the canonically conjugate field \( \pi^{(x)}(\eta, \bar{x}) \)

\[
\pi^{(x)}(\eta, \bar{x})
\]

to the field \( X(\eta, x) \), i.e., the Legendre transform of \( X \).

* To this end, we consider the Lagrangian:

\[
L = \int \frac{1}{2} \left( \dot{X}^2 - \sum_{i=1}^{3} \dot{x}_i^2 - (m^2 - \frac{a^2}{\alpha}) X \right) d^3 x
\]

* Thus, the Legendre transformed variable reads:

\[
\pi^{(x)}(\eta, \bar{x}) = \frac{\partial L}{\partial \dot{X}(\eta, \bar{x})} = X'(\eta, \bar{x})
\]  \( \text{(Eqn 2)} \)

* Which is the field that is conjugate to \( \phi \)?

\[
S_{\phi \phi} = \int \left( \frac{1}{2} \alpha^2(\eta) \phi \phi \phi_{, \alpha} \phi_{, \beta} - \frac{1}{2} m^2 \phi^2 \right) d^4 \eta d^3 x
\]

\( \Rightarrow \) The field \( \pi^{(\phi)} \) which is conjugate to \( \phi \) reads:

\[
\pi^{(\phi)} = \frac{\delta L}{\delta \phi'} = \alpha^2 \phi'
\]

* Compare:

\[
\pi^{(x)} = X' = (a \phi)'
\]

\[
= a \phi' + a' \phi
\]

\[
= \frac{1}{a} \pi^{(\phi)} + a' \phi
\]

\( \text{i.e., } \pi^{(\phi)}, \pi^{(x)} \text{ are different!} \)
$\Box$ **Quantization:**

\[ [\hat{\phi}(q, \bar{x}), \hat{\pi}^{(\phi)}(q, \bar{x}')] = i\delta^3(\bar{x} - \bar{x}') \]

\[ [\hat{\phi}(q, \bar{x}), \hat{\phi}(q, \bar{x}')] = 0 \]

\[ [\hat{\pi}^{(\phi)}(q, \bar{x}), \hat{\pi}^{(\phi)}(q, \bar{x}')] = 0 \]

$\Box$ **Proposition:**

In terms of the fields $\hat{\mathcal{X}} = a \hat{\phi}$, $\hat{\pi}^{(\mathcal{X})} = \hat{\mathcal{X}}'$, these commutation relations become:

\[ [\hat{\mathcal{X}}(q, \bar{x}), \hat{\pi}^{(\mathcal{X})}(q, \bar{x}')] = i\delta^3(\bar{x} - \bar{x}') \]

\[ [\hat{\mathcal{X}}(q, \bar{x}), \hat{\mathcal{X}}(q, \bar{x}')] = 0 \]

\[ [\hat{\pi}^{(\mathcal{X})}(q, \bar{x}), \hat{\pi}^{(\mathcal{X})}(q, \bar{x}')] = 0 \]

$\Box$ **Proof** Only the first CCR is nontrivial to check:

\[ [\hat{\mathcal{X}}(q, \bar{x}), \hat{\pi}^{(\mathcal{X})}(q, \bar{x}')] = [a(q) \hat{\phi}(q, \bar{x}), \frac{1}{a(q)} \hat{\pi}^{(\phi)}(q, \bar{x}') a(q) \hat{\phi}(q, \bar{x}')] \]

\[ = [\hat{\phi}(q, \bar{x}), \hat{\pi}^{(\phi)}(q, \bar{x}')] \]

\[ = i\delta^3(\bar{x} - \bar{x}') \]

$\Box$ **Thus, the change from $\phi$ to $\mathcal{X}$ is fairly trivial.**

Notice, however:

\[
L \xrightarrow{\text{L.T. $\phi$ replaced by $x'$}} H^{(\phi)} := \int \phi \pi^{(\phi)} d^3x - L \quad \text{← they have no reason to be the same!}
\]

\[
\xrightarrow{\text{L.T. $x'$ replaced by $x$}} H^{(\mathcal{X})} := \int x \pi^{(\mathcal{X})} d^3x - L
\]
Question:

How can both be valid generators of time evolution, i.e., how can we have:

\[ i \dot{\phi}' = [\phi', \hat{H}^{(\theta)}] \quad \text{and} \quad i \dot{x}' = [\hat{x}', \hat{H}^{(\theta)}] \]

and yet \( \hat{H}^{(\theta)} \neq \hat{H}^{(0)} \)?

Should there not be one Hamiltonian for all variables?

Answer: Yes, and it is, of course \( \hat{H}^{(\theta)} \).

Recall that in quantum mechanics:

\[ i \dot{Q} = [\hat{Q}, \hat{H}] + i \frac{\partial}{\partial t} \hat{Q} \]

Explicitly:

* From \( \dot{x} = \frac{1}{\alpha} \hat{x} \) and \( i \dot{\phi}' = [\hat{\phi}', \hat{H}^{(\theta)}] \) we obtain:

\[ i \frac{1}{\alpha} \dot{x}' = \frac{1}{\alpha} [\hat{x}, \hat{H}^{(\theta)}] \]

\[ \Rightarrow \quad i \frac{1}{\alpha} \dot{x}' - i \frac{\alpha}{\alpha t} \dot{x} = \frac{1}{\alpha} [\hat{x}, \hat{H}^{(\theta)}] \]

\[ \Rightarrow \quad i \dot{x}' = [\hat{x}, \hat{H}^{(\theta)}] + i \frac{\alpha}{\alpha t} \dot{x} \]

* But we also have:

\[ i \dot{x}' = [\hat{x}, \hat{H}^{(\theta)}] \]

\[ \Rightarrow \quad \text{We must have:} \quad \hat{H}^{(\theta)} = \hat{H}^{(\theta)} \]
Since there are multiple Hamiltonians, which, if anyone, is the energy?

- One usually defines the energy as the generator of time evolution. We saw that in the presence of gravity this is ambiguous: one can define many different Hamiltonians for the same theory (same action).

- Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:

  - Recall: The Einstein equation

    \[ R_{\mu \nu}(x) - \frac{1}{2} g_{\mu \nu}(x) R(x) + \Lambda g_{\mu \nu}(x) = 8\pi G T_{\mu \nu}(x) \]

- Recall: The K.G. field's energy-momentum tensor

  \[ T_{\mu \nu}(\varphi, \pi) = \frac{2}{\hbar^2} \left( \frac{\partial S}{\partial \varphi} \right) = 2\pi_{\mu \nu} - g_{\mu \nu} \left( \frac{1}{2} \sum_{i} \hat{\phi}_i^2 + \frac{1}{2} m^2 \delta_{\mu \nu} \right) \]

- Consider \( T_{00}(\varphi, \pi) \), which is called the "energy density":

  \[ T_{00}(\varphi, \pi) = \frac{1}{4} \frac{\partial S}{\partial \varphi}^2 + \frac{1}{4} \sum_{i} \hat{\phi}_i^2 + \frac{1}{2} \frac{\partial S}{\partial \phi}^2 \]

  (T)

- **Exercises:**

  - a) Verify (T).

  - b) Calculate \( H^{(0)} \).

  - Notice that \( H^{(0)} \) is not a scalar.

  - c) Show that \( H^{(0)}(\varphi) = \int_{\mathbb{R}^3} T^{00}(\varphi, \pi) \sqrt{g} \, d^3x \).

  - d) Calculate \( H^{(0)}(\varphi) \).