QFT for Cosmology, Achim Kempf, Winter 16, Lecture 15

Solving the quantized K.G. eqn. on FRW spacetimes

Recall:

1.) We obtain the solution $\hat{\phi}(x,t)$ through the ansatz

$$\hat{\phi}(x,t) = \sum_k u_k(x,t)\, a_k + u_k^*(x,t)\, a_k^+ \qquad (*)$$

* if we use operators a_k obeying $[a_k, a_{k'}^+] = \delta^3(k-k')$ and
* if we find classical solutions $\{u_k(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\sqrt{|g|}\, g^{0\nu} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^\nu} u_k(x',t) \right) = i\, \delta^3(\vec{x} - \vec{x}') \qquad (C)$$

2.) Then, we can use the $\{a_k\}$ to build a convenient basis in the Hilbert space:

- Namely: $|0\rangle$ is the vector obeying $a_k |0\rangle = 0$
- The other basis vectors are:

$$a_k^+ |0\rangle, \ldots \frac{1}{\sqrt{n!}} (a_k^+)^n |0\rangle, \ldots a_k^+ a_{k'}^+ |0\rangle, \ldots$$

$$\frac{1}{\sqrt{n_1!}} \cdots \frac{1}{\sqrt{n_m!}} (a_{k_1}^+)^{n_1} \cdots (a_{k_m}^+)^{n_m} |0\rangle, \ldots \text{ etc.}$$

3.) Choosing a different set of classical solutions $\{\tilde{u}_k(x,t)\}$ which obey (C) yields the same $\hat{\phi}(x,t)$, namely

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t)\, \tilde{a}_k + \tilde{u}_k^*(x,t)\, \tilde{a}_k^+$$

but the basis of vectors $|\tilde{0}\rangle$, $\tilde{a}_k^+ |\tilde{0}\rangle$, $\tilde{a}_k^+ \tilde{a}_{k'}^+ |\tilde{0}\rangle \ldots$ is a different basis. Recall: Stone von Neumann theorem.

Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat{\phi}(x,t)$ directly, but instead for:

$$\hat{\chi}(\eta,x) := a(\eta)\,\hat{\phi}(\eta,x)$$

- In terms of $\hat{\chi}(\eta,x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta,x) - \Delta\hat{\chi}(\eta,x) + \left(m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{\chi}(\eta,x) = 0$$

- **Note:** This is a partial differential equation because both time and space derivatives occur.

- Observation: The derivatives $\frac{\partial}{\partial x^i}$ become multiplication operators ik_i under spatial Fourier transform.

- Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more managable set of ordinary DEs.

- Define: $\hat{\chi}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}}\,\hat{\chi}(\eta,x)\,e^{-ikx}\,d^3x$

  i.e.: $\hat{\chi}(\eta,x) = \int \frac{1}{(2\pi)^{3/2}}\,\hat{\chi}_k(\eta)\,e^{ikx}\,d^3k$

- Analogously:

$$\hat{\pi}_k^{(\chi)}(\eta) := \int \frac{1}{(2\pi)^{3/2}}\,\hat{\pi}^{(\chi)}(\eta,x)\,e^{-ikx}\,d^3x$$

Thus, in terms of $\hat{x}_k(\eta)$, the K.G. eqn. reads:

$$\boxed{\hat{x}_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\hat{x}_k(\eta) = 0} \qquad \text{(EoM)}$$

$\Rightarrow$ for each comoving Fourier mode k the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\eta) + \omega_k^2(\eta)\,\hat{x}(\eta) = 0$$

$$\text{with:} \quad \omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

Remark: It is not surprising that the frequency of each comoving mode is changing because its physical wavelength is changing to.

In extreme cases:

The frequency $\omega_k(\eta)$ may become imaginary, namely if $a''(\eta)$ is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on k, i.e., some modes may have imaginary frequencies while others don't.

Exercise:

* Show that $\hat{x}_k^+(\eta) = \hat{x}_{-k}(\eta)$, $\hat{\pi}_k^{(x)+}(\eta) = \hat{\pi}_{-k}^{(x)}(\eta)$ (HC)

* Show that

$$[\hat{x}_k(\eta), \hat{\pi}_{k'}(\eta)] = i\,\delta^3(k+k') \qquad \text{(CCR)}$$

$$\text{i.e.} \quad [\hat{x}_k(\eta), \hat{\pi}_{k'}^+(\eta)] = i\,\delta^3(k-k')$$

- In order to solve EoM, HC, CCR for $\hat{x}_k(\eta)$, we we make this ansatz:

$$\hat{x}_k(\eta) := \underbrace{\frac{1}{\sqrt{2}}}_{\text{convenient later}} \left(v_k^*(\eta)\, a_k + v_{-k}(\eta)\, a_{-k}^+ \right) \qquad (A)$$

- **Exercise:** Express the mode functions $u_k(\eta, x)$ of $(*)$ in terms of the functions $v_k(\eta)$.

- **Proposition:** The ansatz (A)

  1.) solves the hermiticity condition (HC) by construction.

  2.) solves the (EoM), if the $v_k(\eta)$ each solve (EoM) as (complex!) number-valued functions:

(**Note:** The equation depends only on $|k|$, not on the direction of k. Thus if $v_k(\eta)$ is a solution for one k then it is solution for all k' with $|k'| = |k|$. $\Rightarrow$ We can and will choose $v_k(\eta) = v_{|k|}(\eta)$)

$$\boxed{v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0 \qquad (M)}$$

  3.) the commutation relations (CCR) if the v_k are chosen such that they also obey:

$$\boxed{v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad (W)}$$

- **Exercise:**

  a) Prove the proposition.

  b.) Assume that $v_k(\eta)$ is any solution of (EoM). Show that if (W) holds at one time then it holds at all time.
  (Note: The LHS of (W) is the "Wronskian" of the ODE)

Conclusion:

In order to obtain the solution $\hat{\phi}(\eta, x)$, we do:

A) Find for each mode $k \in \mathbb{R}^3$ a solution $v_k(\eta)$ to **(M)**, i.e., a solution to the classical harmonic oscillator with time-dependent frequency.

B) Make sure $v_k(\eta)$ obeys **(W)**, if need be by multiplying with a constant. (Recall exercise b))

C) Build a basis in the Hilbert space:

$a_k |0\rangle = 0, \quad a_k^+ |0\rangle, \quad a_k^+ a_{k'}^+ |0\rangle$, etc ...

Choice of mode solutions $\{v_k(\eta)\}$

- For each choice, say $\{v_k(\eta)\}_{k \in \mathbb{R}^3}$ or $\{\tilde{v}_k(\eta)\}_{k \in \mathbb{R}^3}$ we obtain the same $\hat{\phi}(x,t)$ but the bases

  $|0\rangle, \; a_k^+ |0\rangle, \; a_k^+ a_{k'}^+ |0\rangle, \dots$

  and

  $|\tilde{0}\rangle, \; \tilde{a}_k^+ |\tilde{0}\rangle, \; \tilde{a}_k^+ \tilde{a}_{k'}^+ |\tilde{0}\rangle, \dots$

  will of course be different.

- We will often find it convenient to use the basis $|0\rangle, \; a_k^+ |0\rangle, \; a_k^+ a_{k'}^+ |0\rangle, \dots$ that comes with one set of mode functions $\{v_k(\eta)\}$ at one time (say initially) and then the basis $|\tilde{0}\rangle, \; \tilde{a}_k^+ |\tilde{0}\rangle, \; \tilde{a}_k^+ \tilde{a}_{k'}^+ |\tilde{0}\rangle, \dots$ of some $\{\tilde{v}_k(\eta)\}$ later.

Why? ☐ In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.

☐ But the observables evolve in time!

$\Longrightarrow$ The meanings of all Hilbert space vectors change over time

$\rightsquigarrow$ We may, e.g., choose a set $\{v_k\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_k\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

☐ How many possible choices of

$$\{v_k(\eta)\}_{k\in\mathbb{R}^3}, \{\tilde{v}_k(\eta)\}_{k\in\mathbb{R}^3}, \{\hat{\tilde{v}}_k(\eta)\}_{k\in\mathbb{R}^3}, \ldots$$

do exist?

☐ Let us consider each mode, k, separately:

☐ The solution space of (M), for fixed k,

$$v_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right) v_k(\eta) = 0$$

has of course 2 complex dimensions.

Note: Every solution obeying (W) must be complex-valued. Why?

☐ If v_k is a complex-valued solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.

$\Longrightarrow$ Every solution, $\tilde{v}_k$, is a linear combination of v_k, v_k^*, i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta)$$

- The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions $v_k, \tilde{v}_k$, etc must also obey (W), i.e.:

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k^*(\eta)' = 2i \qquad \text{(W)}$$

(i.e. $\mathrm{Im}\,(v' v^*) = 1$, which is only one real equation)

- **Proposition:**

Assume v_k obeys (W). Then, $\tilde{v}_k$ defined through

$$\tilde{v}_k(\eta) = \alpha_k\, v_k(\eta) + \beta_k\, v_k^*(\eta) \qquad \text{(B)}$$

also obeys (W), iff the coefficients α_k, β_k obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

- Proof: Exercise

$\Rightarrow$ we easily obtain:

$$\left.\begin{aligned} \hat{x}_k(\eta) &= \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right) \\ &= \frac{1}{\sqrt{2}}\left(\tilde{v}_k^*(\eta)\, \tilde{a}_k + \tilde{v}_k(\eta)\, \tilde{a}_{-k}^+\right) \end{aligned}\right\} \text{(P)}$$

$$= \ldots$$

Terminology: Such a transformation from one choice $\{v_k\}, a_k$ and corresponding basis

$$|0\rangle,\ a_k^+|0\rangle,\ a_k^+a_{k'}^+|0\rangle,\ \dots$$

to some $\{\tilde{v}_k\}, |\tilde{0}\rangle, \tilde{a}_k$ and their basis

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|\tilde{0}\rangle,\ \tilde{a}_k^+\tilde{a}_{k'}^+|\tilde{0}\rangle,\ \dots$$

is called a "**Bogolubov transformation**".

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.
* Find out when which choice of $\{v_k\}$ is convenient. (E.g. so that $|0\rangle$ is, at least at one time, the vacuum.)

Bogolubov transformations of Hilbert bases

- How can we express the basis vectors

$$|\tilde{0}\rangle,\ \tilde{a}_k^+|\tilde{0}\rangle,\ \frac{1}{\sqrt{2!}}\tilde{a}_k^{+2}|\tilde{0}\rangle,\ \dots,\ \tilde{a}_k^+\tilde{a}_{k'}^+|\tilde{0}\rangle,\ \dots$$

as linear combinations of the basis

$$|0\rangle,\ a_k^+|0\rangle,\ \frac{1}{\sqrt{2!}}a_k^{+2}|0\rangle,\ \dots,\ a_k^+a_{k'}^+|0\rangle,\ \dots\ ?$$

- **Proposition:** Equations (B) & (P) yield: $\boxed{a_k = \alpha_k^*\tilde{a}_k + \beta_k\tilde{a}_{-k}^+}$

Proof: Exercise.

- Now we observe that $a_k|0\rangle = 0$ becomes:

$$(\alpha_k^*\tilde{a}_k + \beta_k\tilde{a}_{-k}^+)|0\rangle = 0$$

- Try to solve for $|0\rangle$ using ansatz: $|0\rangle := \left(\prod_k f_k(\tilde{a}^+_k, \tilde{a}^+_{-k})\right)|\tilde{0}\rangle$
- Proposition:

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k^*}\tilde{a}^+_k \tilde{a}^+_{-k}}\right]|\tilde{0}\rangle \quad (T)$$

needed for normalization

- Proof: Exercise.
  Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \ldots$

Interpretation of (T):

- Assume, e.g., that $|0\rangle$ and $|\tilde{0}\rangle$ are those Hilbert space vectors which happen to be the vacuum state vectors at the times η_1 and η_2 respectively.

(We will soon explore how to identify the vacuum state at any given time)

- Assume at time η_1 the system's state $|\Omega\rangle$ is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions $\{v_k\}$ so that:

$$|\Omega\rangle = |0\rangle$$

At a later time, η_2, the system is still in this state but the vacuum is then some other vector $|\tilde{0}\rangle$, which obeys $\tilde{a}_k|\tilde{0}\rangle = 0$ with mode functions $\tilde{v}_k(t)$.

- Thus, from (T) we see that the system's state, $|\Omega\rangle$, is at η_2 a state with many particles.
- Note: the particles have been created in $k, -k$ pairs.
- Intuitively: The expansion rips virtual particle + antiparticle pairs apart.