

From the particle picture to the wave picture

So far: Spacetime dynamics can produce particles.

When? When mode oscillators $\omega_k(t)$ changes nonadiabatically fast: $\omega_k(t)' / \omega_k(t)^2 \gg 1$.

In cosmology? No: particles mostly produced conventionally from inflation potential at the end of inflation.

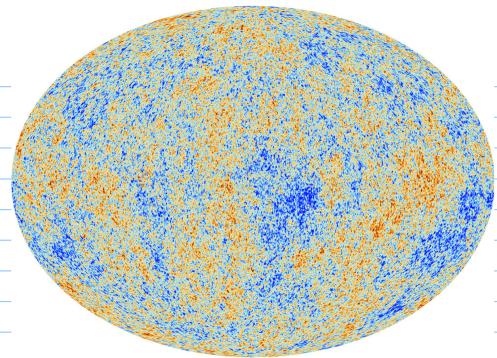
Now: Spacetime dynamics can enhance quantum field fluctuations!

When? When $\omega_k(\eta)$ becomes imaginary.

Recall: $\omega_k^2(\eta) := k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$

Quantum field fluctuations

Do the amplitudes $\hat{\phi}(x, t)$ of a quantum field necessarily quantum fluctuate?



Yes: \square Consider a real-valued function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ and a time η_0 .

\square Then, we define the state $|f\rangle$ as the joint eigenstate of all operators $\hat{\phi}(x, \eta_0)$ with eigenvalues $f(x)$:

$$\hat{\phi}(x, \eta_0) |f\rangle = f(x) |f\rangle$$

\square There are no quantum fluctuations of the quantum field $\hat{\phi}$ if the system is in such a state $|f\rangle$:

Expectation value: $\bar{\phi}(x, \eta_0) = \langle \int | \hat{\phi}(x, \eta_0) | f \rangle$
 $= f(x) \langle \int | f \rangle$
 $= f(x)$

Variance: $\Delta\phi^2(x, \eta_0) = \langle \int | (\hat{\phi}(x, \eta_0) - \bar{\phi}(x, \eta_0))^2 | f \rangle$
 $= \langle \int | (f(x) - f(x))^2 | f \rangle$
 $= 0$ i.e. no fluctuations.

But, can such states $|f\rangle$ occur in practice?

No! The reason is that for these states:

$$\langle \int | \hat{H}^{(0)}(\eta_0) | f \rangle = \infty \quad \text{Exercise: Show this.}$$

Hint: For these states, $\Delta\phi = 0$, and so $\Delta\pi^2 = \infty$

But $\hat{H}^{(0)}$ contains a term π^2 ...

Exercise:

What is the analogue of this observation in the case of the harmonic oscillator in quantum mechanics?

\Rightarrow * Even the state $|f\rangle$ with $f(x)=0$ for all x has ∞ energy and is, therefore, not accessible.

* Thus, all finite energy states do possess quantum fluctuations

How to calculate the amount of quantum field fluctuations?

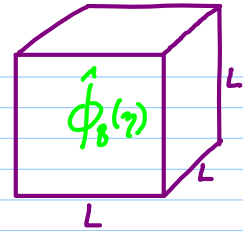
□ It is not realistic to measure all operators $\hat{\phi}(x)$ individually.

□ Realistically, we could at most hope to measure an average of the values of $\hat{\phi}$ over regions $B \subset \mathbb{R}^3$ of not too small volume $V = L \times L \times L$:

$$\hat{\phi}_B(\eta) := \int_{\mathbb{R}^3} \hat{\phi}(x, \eta) W(x) d^3x$$

with "window function" W

$$W(x) = \begin{cases} \approx 0 & \text{for all } x \notin B \\ \approx V^{-1} & \text{for all } x \in B \end{cases}$$



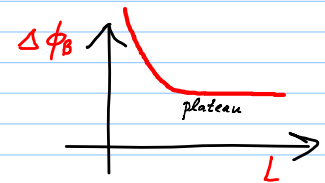
(we'll also allow B to be spherical)

Plan:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size.



- Calculate how this relationship is affected by cosmic expansion.



Quantum field fluctuations in FRW spacetime

- Choose conformal time η and comoving coordinates x .
- Choose a region B of size $L \times L \times L$.

Note: In proper coordinates, this is a box of increasing size:

$$a(\eta)L \times a(\eta)L \times a(\eta)L$$

- Assume that at η_0 the system's state, $|\Omega\rangle$, is the vacuum state:

$$|\Omega\rangle = |\text{vac}_{\eta_0}\rangle$$

- We choose the mode functions $v_k(\eta)$ so that $|\text{vac}_{\eta_0}\rangle = |0\rangle$ with:

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}} (v_k^*(\eta) a_k + v_k a_{-k}^\dagger) \text{ and } a_k |0\rangle = 0$$

→ The expectation value of the region-averaged field at a time $\eta \gg \eta_0$:

$$\begin{aligned}\bar{\phi}_B(\eta) &= \langle \Omega | \hat{\phi}_B(\eta) | \Omega \rangle \\ &= \langle \text{vac}_{\eta_0} | \hat{\phi}_B(\eta) | \text{vac}_{\eta_0} \rangle \\ &= \langle 0 | \int_{\mathbb{R}^3} \hat{\phi}(x, \eta) W(x) d^3x | 0 \rangle \\ &= \langle 0 | \frac{1}{a(\eta)} \int_{\mathbb{R}^3} \hat{\chi}(x, \eta) W(x) d^3x | 0 \rangle \\ &= \frac{1}{a(\eta)} \int_{\mathbb{R}^3} \langle 0 | \frac{1}{\sqrt{2}} (\cancel{v_i^+(\eta) a_k} + v_k(\eta) \cancel{a_k^+}) | 0 \rangle \\ &= 0\end{aligned}$$

→ The average amplitude of $\hat{\phi}_B$ vanishes in the vacuum state.

□ The quantum fluctuations

While $\bar{\phi}_B(\eta)$ vanishes, measurement outcomes for $\hat{\phi}_B(\eta)$ are not fully predictable because subject to fluctuations around zero with this standard deviation:

$$\begin{aligned}\Delta \phi_B^2(\eta) &= \langle \Omega | (\hat{\phi}_B(\eta) - \bar{\phi}_B(\eta))^2 | \Omega \rangle \\ &= \langle 0 | \hat{\phi}_B(\eta)^2 | 0 \rangle \\ &= \frac{1}{a(\eta)^2} \langle 0 | \left(\int_{\mathbb{R}^3} \hat{\chi}(x, \eta) W(x) d^3x \right)^2 | 0 \rangle \\ &= \dots \text{ Exercise: fill in the steps}\end{aligned}$$

$$= \frac{1}{2a(\gamma)^2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |V_k(\gamma)|^2 |\tilde{W}(k)|^2 d^3k$$

↑
Fourier transform of the window function $W(x)$.

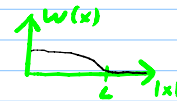
Assume now for simplicity that B is spherical with radius L . Then use spherical coordinates:

$$= \frac{1}{2a(\gamma)^2} \frac{1}{(2\pi)^3} \int_0^\infty k^2 4\pi |V_k(\gamma)|^2 \tilde{W}(k) dk$$

↑ $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$

Notice the dimension dependence of the integral's measure!

Approximation: Consider that:



↳ typical scale is L

(using Fourier) \Rightarrow We can assume that, roughly:

$$\tilde{W}(k) \approx 0 \text{ for } |k| > \frac{2\pi}{L}$$

Example: If $W(x) = \text{rect}_L(x)$, then $\tilde{W}(k) = \frac{\sin(kL)}{kL}$

and we approximate that $\tilde{W}(k) \approx$

$$\Rightarrow \Delta \phi_B^2(\gamma) \approx \frac{1}{4\pi^2 a(\gamma)^2} \int_0^{2\pi/L} k^2 |V_k(\gamma)|^2 dk$$

In the integral, the values of $|v_k(\eta)|^2$ for small k are suppressed by k^2 .

\Rightarrow Can approximately replace $|v_k(\eta)|$ by its value at $k = 2\pi/L$:

$$\Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2} \int_0^{2\pi/L} k^2 |v_{\frac{2\pi}{L}}(\eta)|^2 dk$$

$$\begin{aligned} \Delta \phi_B^2(\eta) &\approx \frac{1}{4\pi^2} \frac{(2\pi)^3}{3L^3} \left| \frac{v_{\frac{2\pi}{L}}(\eta)}{a(\eta)} \right|^2 \\ &= \frac{2\pi}{3L^3} |w_{\frac{2\pi}{L}}(\eta)|^2 \end{aligned}$$

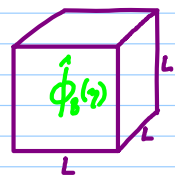
Here, w is the mode function of $\hat{\phi}$, because $\frac{\dot{\chi}(\eta)}{a(\eta)} = \hat{\phi}(\eta)$.

Conclusion:

Assume that at a time η_0 the vacuum state was the state $|0\rangle$ corresponding to $\{v_k\}$.

Then, the typical amplitude of fluctuations of size L at an arbitrary time η is:

$$\Delta \phi_B^2(\eta) \approx \frac{2\pi}{3L^3} \left| \frac{v_{\frac{2\pi}{L}}(\eta)}{a(\eta)} \right|^2 = \frac{2\pi}{3L^3} |w_{\frac{2\pi}{L}}(\eta)|^2$$



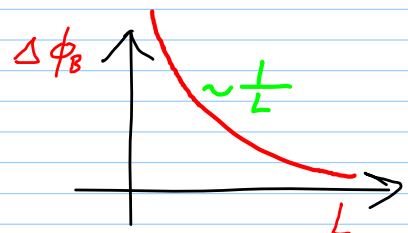
Special case: Minkowski space (massless field)

$$\square v_k(\eta) = w_k(\eta)$$

$$\square \eta = t$$

$$\square |v_k(t)|^2 = \frac{1}{\omega_k(t)} = \frac{1}{|k|} = \frac{L}{2\pi}$$

$$\Rightarrow \Delta \phi_B^2 \approx \frac{1}{3L^2} \quad \Rightarrow \Delta \phi_B \approx \frac{1}{\sqrt{3}} \frac{1}{L}$$



How to describe field quantum fluctuations using correlators

A primer on classical fluctuations:

□ Assume $n(t)$ is a Ω -bandlimited gaussian white noise signal, i.e., a random signal with gaussian distributed amplitudes, filtered to leave only frequencies in the interval $[-\Omega, \Omega]$.

all frequencies occur to same amount

□ Then, for an ensemble of such noise signals, one can show:

$$\overline{n(t)} = 0 \quad \forall t$$

$$\text{"2-point correlator": } \overline{n(t)n(t+L)} = c \frac{\sin(\Omega L)}{\Omega L} \quad \forall t$$

How can we see this?

This noise is ergodic, i.e. we could instead average over all t :

$$\begin{aligned} \overline{n(t)n(t+L)} &= \int f(t)f(t+L)dt \quad \left(\text{suitably regularize if non-normalizable, e.g. } \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T f(t)f(t+L)dt \right) \\ \text{"Auto-correlator"} &= \iiint \tilde{f}(\omega)\tilde{f}(\omega')e^{i\omega t}e^{i\omega'(t+L)}dtd\omega d\omega' \\ &= \iiint \underbrace{e^{it(\omega+\omega')}}_{=(2\pi)^{-1}\delta(\omega+\omega')}dtd\tilde{f}(\omega)\tilde{f}(\omega')e^{i\omega'L}d\omega d\omega' \\ &= \frac{1}{2\pi} \int \tilde{f}(\omega)\tilde{f}(-\omega)e^{i\omega L}d\omega \\ &= \frac{1}{2\pi} \int |\tilde{f}(\omega)|^2 e^{i\omega L}d\omega \quad \rightarrow \text{"Spectral power function"} \end{aligned}$$

⇒ □ Auto correlation and power spectrum are a Fourier pair!

Recall: flatness of spectrum means noise is "white"

□ For white bandlimited noise: $|\tilde{f}(\omega)|^2 = \begin{matrix} \downarrow & \downarrow \\ \text{rect} & \text{rect} \\ \left[-\frac{\Omega}{2}, \frac{\Omega}{2} \right] & \left[-\frac{\Omega}{2}, \frac{\Omega}{2} \right] \\ \omega & \omega \end{matrix}$

Exercise: Show that its Fourier transform is indeed $\sin(\Omega L)/\Omega L$.

The 2-point correlator in QFT:

$$\langle 0 | \hat{\phi}(\vec{x}, \eta) \hat{\phi}(\vec{x} + \vec{L}, \eta) | 0 \rangle = \frac{1}{a(\eta)^2} \langle 0 | \hat{\chi}(\vec{x}, \eta) \hat{\chi}(\vec{x} + \vec{L}, \eta) | 0 \rangle$$

Exercise: use mode expansion of $\hat{\chi}$ and spherical coordinates to derive:

$$= \frac{1}{a(\eta)^2} \int_0^\infty \frac{k^2 dk}{4\pi^2} \frac{\sin(kL)}{kL} |v_k(\eta)|^2 \quad \text{with } k = |\vec{k}|, L = |\vec{L}|.$$

↑ Notice dimension dependence of the integral's measure!

Observe: $\frac{\sin(kL)}{kL} \approx \begin{cases} 1 & \text{for } k < \frac{1}{L} \\ 0 & \text{for } k > \frac{1}{L} \end{cases}$

Observe: $k^2 \frac{\sin(kL)}{kL}$ has largest amplitude around $k \approx \frac{2\pi}{L}$

⇒ Estimate:

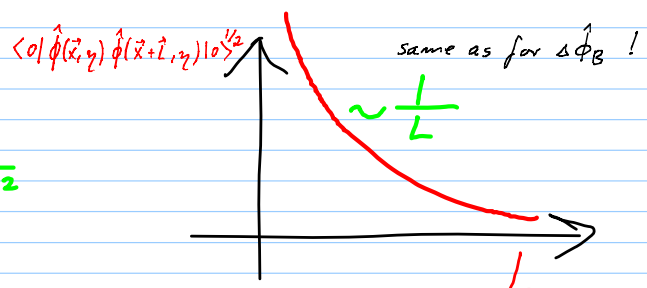
$$\langle 0 | \hat{\phi}(\vec{x}, \eta) \hat{\phi}(\vec{x} + \vec{L}, \eta) | 0 \rangle \approx a(\eta)^{-2} \int_0^{2\pi/L} \frac{k^2}{4\pi^2} |v_{\frac{k}{2}}(\eta)|^2$$

$$\approx a(\eta)^{-2} \frac{k^3}{12\pi^2} |v_k(\eta)|^2 \Big|_{k=\frac{2\pi}{L}}$$

Special case: Minkowski space

Mode function: $|v_k|^2 = \frac{1}{|k|}$

⇒ $\langle 0 | \hat{\phi}(\vec{x}, \eta) \hat{\phi}(\vec{x} + \vec{L}, \eta) | 0 \rangle \approx \frac{1}{3} \frac{1}{L^2}$



We notice: The variance in a box scales like the correlator!
Both are good measures of the fluctuations.

Definition: We define the so-called Fluctuation Spectrum at time η as a function of k :

$$\delta\phi_k(\eta) := a(\eta)^{-1} k^{3/2} |v_k(\eta)|$$

$$k = \frac{2\pi}{L}$$

Special case Minkowski space with massive field:

□ Scale factor: $a(\eta) = 1$ for all η

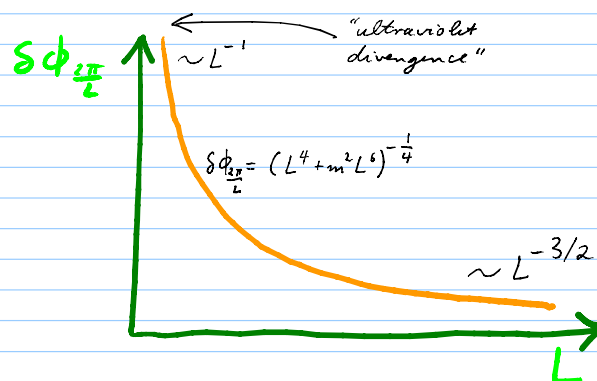
□ Mode functions:

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\eta\omega_k} \quad \text{with } \omega_k = \sqrt{k^2 + m^2}$$

⇒ The fluctuation spectrum is: (recall: $k = \frac{2\pi}{L}$)

$$\delta\phi_k = \frac{k^{3/2}}{(m^2 + k^2)^{1/4}} \approx \begin{cases} k & \text{for } k \rightarrow \infty \\ \frac{k^{3/2}}{\sqrt{m}} & \text{for } k \rightarrow 0 \end{cases}$$

and as a function of L it is: $\delta\phi_{\frac{2\pi}{L}} = \frac{(2\pi)^{3/2} L^{-3/2}}{(\frac{4\pi^2}{L^2} + m^2)^{1/4}}$



Notice: The two different scaling behaviors are not clearly visible in this plot.

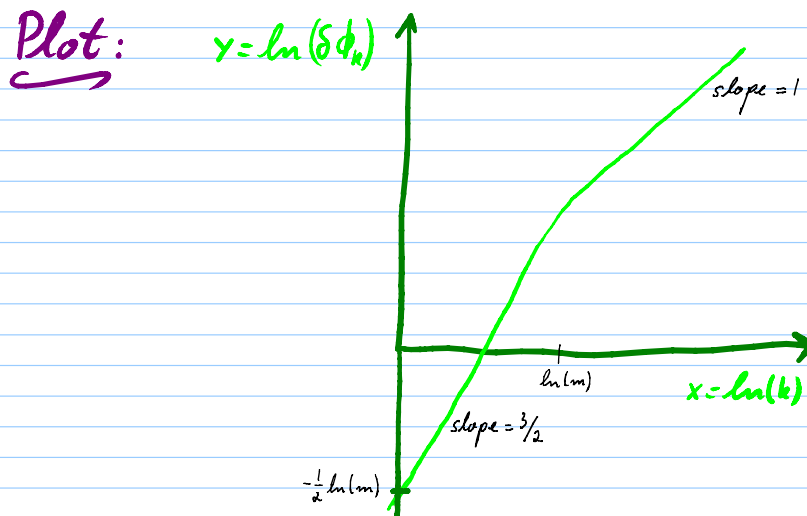
Recall "Log-Log plots":

$$x := \ln(k), \quad y = \ln(\delta\phi_k)$$

Here:
$$\ln \delta\phi_k = \ln\left(\frac{k^{3/2}}{(m^2+k^2)^{1/4}}\right) \approx \begin{cases} \ln k & \text{for } k \rightarrow \infty \\ \ln\left(\frac{k^{3/2}}{\sqrt{m}}\right) & \text{for } k \rightarrow 0 \end{cases}$$

$$\underbrace{\ln\left(\frac{k^{3/2}}{\sqrt{m}}\right)}_{= -\frac{1}{2}\ln(m) + \frac{3}{2}\ln k}$$

Thus:
$$y \approx \begin{cases} x & \text{for } x \rightarrow \infty \\ -\frac{1}{2}\ln(m) + \frac{3}{2}x & \text{for } x \rightarrow -\infty \end{cases}$$



□ We notice that, in Minkowski space, large scale (i.e., large L , small k) fluctuations are strongly suppressed, especially if mass $m \neq 0$.

□ Regarding the Infrared (IR):

- * The mass m does not matter at very short distances, i.e., in the ultraviolet.
- * But, for large L the mass term does help to suppress the quantum fluctuations. ($\delta\phi \sim L^{-3/2}$ vs. $\sim L^{-1}$)
- * Generally, in phenomena of QFT, the mass of particles tends to play a rôle only in the infrared, but not in the ultraviolet, which is why in studies of UV phenomena the mass can often be neglected (e.g. for "perturbative power counting" - a method you encounter in renormalization).

□ Regarding the Ultraviolet (UV)

- * We see that QFT predicts divergingly large quantum fluctuations to be found in measurements that resolve smaller and smaller regions.
 - * For small enough regions B the fluctuations $\Delta\phi_B$ in ϕ_B would lead to fluctuations in the Klein Gordon energy momentum tensor that are large enough to cause black holes.
- ⇒ At this scale, $\approx 10^{-35} m$, the Planck scale, the notion of distance is assumed to break down.
(Accelerators can probe only distances down to about $10^{-15} m$)