Time evolution and the fluctuation spectrum:

Recall:
- We assume the system is in the state \( |0\rangle \) which is the vacuum at \( \eta_0 \).
- \( \Rightarrow \) The system is always in the state \( |0\rangle \) (Heisenberg picture).
- We solve the QFT with \( \hat{F}_k(\eta) = a(\eta) \hat{\phi}_k(\eta) \) and the ansatz

\[
\hat{C}_k(\eta) = \frac{1}{\sqrt{2}} \left( v_k^+(\eta) a_k + v_k(\eta) a_k^+ \right)
\]

when for convenience we choose the mode functions \( v_k(\eta) \), so that \( a_0 |0\rangle = 0 \).
- The technical challenge will be:
  - Identify \( |0\rangle \), i.e., identify the initial conditions for the \( V_k \) at \( \eta_0 \).
  - Solve the K.G. eqn for the \( V_k(\eta) \).

Benefit:
- State \( |0\rangle \) known
- Operators \( \hat{F}_k(\eta) \) known \( \forall \eta > \eta_0 \).

\( \Rightarrow \) We can calculate all predictions for all times, even, e.g., at times of nonadiabaticity or inverted potential!

In particular, we can calculate for all \( \eta > \eta_0 \):

\[
\delta \phi_k(\eta) = k^{3/2} \frac{v_k(\eta)}{a(\eta)}
\]

We observe: The dynamics of \( |v_k(\eta)| \) crucially affects \( \delta \phi_k(\eta) \).

Q: In which circumstances does \( v_k(\eta) \) grow most?
Answer: The most efficient mechanism to enlarge $\phi$ occurs when the mode is nonadiabatically evolving in the sense that the mode oscillator is inverted:

$$\frac{d^2}{dx^2} \psi(x) + \omega^2 \psi(x) = 0$$

In such a time period, the Klein-Gordon equation's solutions are not oscillatory because $\omega(x)$ is imaginary:

$$\omega(x) = \sqrt{k^2 + m^2} \left| a(x) \right|^2 - \frac{\omega'(x)}{a(x)}$$

This term may be large enough to make the discriminant negative.

* Instead, there will be one exponentially decaying and one exponentially growing solution. Inverting a harmonic oscillator is an efficient way to increase $\Delta \phi$:

\[ \text{Notice that this argument applies to } \psi \text{ limit } \phi = \frac{1}{\alpha} \psi. \]

Thus, $\phi$'s growth is slower than that of $\psi$.

Recall: The equation of motion of $\phi$ has a friction-type term.
Before we calculate the fluctuation amplification explicitly:

**Relationship of fluctuation amplification to particle creation**

1. Assume that at a later time, \( \eta \), the evolution is adiabatic for mode \( k \) (i.e., its \( u_k \) changes slowly).

   \[ \Rightarrow \text{We can identify} \ |\text{vac}_\eta> \]

   Using the adiabatic vacuum identification criterion, we find the mode function \( \tilde{v}_k \) for which:

   \[ |\tilde{\eta}> = |\text{vac}_\eta> \]

2. **Case 1:** The evolution of mode \( k \) was adiabatic from \( \eta_0 \) to \( \eta \).

   * Therefore:

   \[ v_k = \tilde{v}_k \quad \text{and} \quad |\tilde{\eta}> \equiv |\tilde{\eta}>(\eta) \]

   * Therefore:

   The state of the system, \( |\tilde{\eta}> \equiv |\tilde{\eta}(\eta)> \), is still the vacuum state at time \( \eta_0 \):

   \[ |\tilde{\eta}(\eta_0)> = |\tilde{\eta}(\eta)> \]

   * There is no particle creation.

   * But since \( u_k = \frac{1}{\sqrt{\omega_k}} e^{i\chi_k} A_k \), in general:

   \[ |V_k(\eta)> \neq |V_k(\eta_0)> \quad \text{namely} \quad |V_k(\eta)> = |V_k(\eta_0)> \]

   \[ \Rightarrow \text{the fluctuations, which depend on} \ |V_k(\eta)> \]

   can be affected even if there is no particle creation.
**Case 2:** The evolution was not always adiabatic between \( \eta \), and \( \eta_i \).

* Then, \( V_\eta \neq \bar{V}_\eta \)

* but since both are in the same 2-dimensional solution space to the K.G. equation, there exist \( d_\beta, \beta_\beta \):

\[
V_\eta(\eta) = d_\beta \bar{V}_\beta(\eta) + \beta_\beta \bar{V}_\eta(\eta)
\]

* Substitute in the fluctuations equation:

\[
\delta \Phi_\beta(\eta) = \alpha^2(\eta) k^3 |V_\eta(\eta)|^2
\]

\[
= \alpha^2(\eta) k^3 |d_\beta \bar{V}_\beta(\eta) + \beta_\beta \bar{V}_\eta(\eta)|^2
\]

* For clarity, assume that the nonadiabatic period is one by \( \eta_i \).

* Also, assume that spacetime is again Minkowski around \( \eta_i \). (Thus, we focus on nonadiabatic effects only)

* In this case:

\[
\bar{V}_\eta(\eta) = \frac{1}{\omega(\eta)} e^{i \omega(\eta) \eta} \quad \text{for all } \eta \approx \eta_i
\]

\[
\Rightarrow \delta \Phi^2(\eta) = \alpha^2(\eta) \frac{k^3}{\omega(\eta)^2} (d_\beta^2 + |\beta_\beta|^2 - 2 \text{Re}(d_\beta \beta_\beta e^{2i \omega(\eta) \eta}))
\]

* We use: \( |d_\beta|^2 - |\beta_\beta|^2 = 1 \) \((W)\)
\[ \delta \phi_k^2(\tau) = a^{-2}(\tau) \frac{k^3}{1 + 2|\beta_1|^2} (1 + 2|\beta_1|^2) \quad (T) \]

\[ \text{This term is only seen if } \text{the evaporation of quark in the reheating.} \]

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\* Notice: |\phi_1| and |\phi_2| can both become very large. This is consistent with the Wronskian condition, (W).

\* Particle production:

Recall that the expected number of created particles is also given by |\phi_k|^2:

\[ \overline{N}_k(\tau) = \langle \Omega | \hat{N}_k | \Omega \rangle = \ldots \]

\[ = |\phi_k|^2 \]

\[ \overline{N}_k = \overline{a}_k^* \overline{a}_k \]

\* Remark:

But (T) holds even if \( w_0 < 0 \) at \( \tau_0 \). We know what we mean by field fluctuations even when we do not have a concept of vacuum and particles.

\[ \text{"Quantum fields more fundamental than quantum particles."} \]

\[ \sum \text{Fluctuations in proper coordinates as opposed to comoving coordinates} \]

\* We have: \( d = a(\tau) L \), \( p = \frac{1}{a(\tau)} k \)

\[ \text{proper length} \quad \text{comoving} \quad \text{proper} \quad \text{comoving} \quad \text{momentum} \quad \text{momentum} \]

\* Therefore, \( S\phi_k^2(\tau) = a^{-2}(\tau) \frac{k^3}{1 + 2|\beta_1|^2} (1 + 2|\beta_1|^2) \) becomes:
\[ \delta \Phi_0(z) = a^{-2} \frac{a^3 p^3}{1 + 2 |p_a|^2} (1 + 2 |p_a|^2) \]

\[ = \frac{p^3}{\sqrt{p^2 + m^2}} \left( 1 + 2 |p_a|^2 \right) \quad \text{(same as earlier)} \]

* Note: The nonadiabatic term depends on \( p \).

* Note: This was the case when we ended in a Minkowski space. We see that in this case we must get back the original Minkowski spectrum if the evolution from \( \eta_0 \) to \( \eta \) was adiabatic.

* Note: Also in thermodynamics, "adiabatic" processes are reversible processes.

**Application to specific cosmological models**

The standard model of cosmology holds that the very early universe underwent a short period of almost exponential expansion, "inflation".

- Begin by studying QFT in de Sitter spacetime:

The de Sitter FRW spacetime can be defined through

\[ \alpha(t) := e^{Ht} \quad \text{for all } t \in \mathbb{R} \]

Here: \( H > 0 \) is a constant, the "Hubble constant".

Exercise: Read Mukhanov's comments on de Sitter space.

Notes: * \( t \) is the time in a conveniently chosen unit which * begins at \( \eta \) of deep acceleration.
The de Sitter horizon

**Proposition:** (in particle picture) (Note: long $H \to$ small horizon $d_{H}$)

Objects (or any observers) who are further apart than a proper distance of $d_{H} = \frac{1}{H}$
can never meet, and cannot communicate.

**Proof:** * Consider an observer in a galaxy $A$. Let us choose the origin of the comoving coordinate system(s)
to be where this observer sits.

* Now suppose that, at some arbitrary time, $t_{s}$, this observer sends a radio signal towards another galaxy, $B$.

* The signal travels in a small time $\Delta t$

the small comoving distance $\Delta x$:

\[
\frac{a(t) \Delta x}{\Delta t} \approx c = 1
\]

\[
\Rightarrow \quad \frac{dx}{dt} = a^{-1}(t) \text{ i.e.: } \frac{dx}{dt} = e^{-Ht}
\]

\[
\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + C'
\]

Fix the integration constant $C'$ so that $x(t_{s}) = 0 \Rightarrow C' = \frac{1}{H} e^{-Ht_{s}}$

[Recall: $x$ is proper distance towards $A$; $d_{S}(t) = a(t) x(t)$]

\[
\Rightarrow \quad x(t) = -\frac{1}{H} e^{-Ht} + \frac{e^{-Ht_{s}}}{H}
\]

\[
\Rightarrow \quad \text{As } t \to \infty \text{ we have } x(t) \to \frac{e^{-Ht_{s}}}{H}.
\]

Thus, any galaxy $B$ if comoving distance is at most $d_{c} = \frac{e^{-Ht_{s}}}{H}$.

**Q:** Proper distance $d_{p}$ of such $B$ from $A$ at $t_{s}$?

**A:** $d_{s} = a(t) d_{c} \Rightarrow d_{s} = e^{+Ht_{s}} \frac{e^{-Ht_{s}}}{H} = \frac{1}{H}$
Recall: This holds for arbitrary $t$. 

$\Rightarrow$ A signal sent by $A$ at any time $t_s$ can only ever reach $B$ if at the time of sending, $t_s$, the proper distance between $A$ and $B$ is at most $\frac{1}{H}$.

$\Rightarrow$ Any two observers further apart than a proper distance of $\frac{1}{H}$ cannot communicate.

Interpretation: In the case where a de Sitter exponential expansion lasts forever, between any objects of proper distance $> \frac{1}{H}$, space is being created faster than what can be crossed when travelling with the speed of light.

Remark: Notice that the proper size of the de Sitter horizon is constant in time.

Proposition: (in wave picture)

Klein Gordon modes oscillate while their proper wavelength obey $\lambda < \frac{1}{H}$ but stop oscillating and possess instead an imaginary frequency when their proper wavelength has grown beyond, i.e., when $\lambda > \frac{1}{H}$, assuming that their mass is small: $m << H$.

Proof: 1) Let us switch to conformal time: (Thus, need $a(\eta)$!)

$\square$ Recall: $\gamma(t) := \int_{a(t')}^t \frac{1}{a(t')} dt'$

here: $\gamma(t) = \int_{a(t')}^t e^{-Ht'} dt'$

$= -\frac{1}{H} e^{-Ht} + C$
Notice:

- As $t \to -\infty$ we have $\eta \to -\infty$.
- But as $t \to +\infty$ we have $\eta \to 0$.

Choose $C' = 0$:

$$\Rightarrow \eta(t) = -\frac{1}{H} a(t)$$

$$a(t) = -\frac{1}{H \eta(t)}$$

i.e.:

$$a(\eta) = -\frac{1}{H \eta}$$

2) Introduce $\hat{X}_v(\eta) := a(\eta) \hat{\phi}_v(\eta)$:

- We have: $\hat{X}_v(\eta) = -\frac{1}{H \eta} \hat{\phi}_v(\eta)$
- $\hat{X}_v$ obeys this Klein Gordon equation

$$\hat{X}_v''(\eta) + \omega_v^2(\eta) \hat{X}_v(\eta) = 0$$

with:

$$\omega_v^2(\eta) = k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$$

Exercise: Show that in the de Sitter case this yields:

$$\omega_v^2(\eta) = k^2 + \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2}$$
3.) Check for imaginary frequencies. \( \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2} < 0 \)

- Recall: We are assuming \( m << H \).

- Thus, in \( \omega_k^2(\eta) = k^2 + \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2} \)
  
  we have: \( \omega_k^2(\eta) < 0 \)

- Therefore: For each mode \( k \) there comes a time when \( \omega_k^2 \) becomes negative!

  The case relevant in cosmology: \( m = 0 \) (we'll assume this)

  \[ \Rightarrow \text{The time when a mode } k \text{ crosses the horizon is given by:} \]

  \[ \eta_{\text{hor}}(k) \approx -\frac{\sqrt{2}}{k} \]

4.) Conclusion:

- A mode oscillates as long as:
  
  Recall: \( \eta \in (-\infty, 0) \)
  
  i.e. \( |\eta| \gg \frac{1}{k} \) means early times.

  \[ |\eta| \approx \frac{1}{k} \]

  i.e., while \( |\eta|k \gg 1 \)  \( \quad \text{(used that } k \text{ and } \eta \text{ are of same order of magnitude)} \)

- A mode has imaginary frequency from when

  \( |\eta| \ll \frac{1}{k} \) i.e., from when \( |\eta|k \ll 1 \)

  This is late time, i.e. when \( \eta \approx 0 \).

- Re-expressed in terms of proper wavelength?

  Noting \( |\eta| = \frac{1}{Ha} \) and multiplying it with \( k \approx 2\pi \) we obtain:

  \[ |\eta|k = \frac{1}{Ha} \frac{2\pi}{L} \]

  \( \text{-comoving wavelength} \)
Transforming to the proper wavelength, \( \lambda = a(q) b \), we obtain:

\[
|q| k = \frac{2\pi}{H \lambda} \quad \text{Thus, the proper wavelength } \lambda, \text{ of } q \text{ fixed}
\]

\[
|q| k = \frac{2\pi}{H |q|} \quad \text{commoving mode } |k| \text{, always:}
\]

Thus, finally, the two cases, (a) and (b) become:

(a) A mode oscillates as long as: \(|q| k \gg 1\)

\[\text{i.e., as long as } \frac{2\pi}{H |q|} \gg 1 \quad \text{i.e.: } \lambda \ll \frac{1}{H} \]

(b) A mode has imaginary frequency when:

\[\text{from when } \lambda \gg \frac{1}{H} \]

This is what we had set out to show.