QFT for Cosmology, Achim Kempf, Winter 16, Lecture 19

Note Title

Recall de Sitter model spacetime:

$$a(t) := e^{Ht}$$

$$\eta(t) = -\frac{1}{H} e^{-Ht}$$

$$a(\eta) = -\frac{1}{H\eta}$$

$$\hat{x}_k''(\eta) + \omega_k^2(\eta)\,\hat{x}_k(\eta) = 0$$

$$\omega_k^2(\eta) = k^2 - \frac{2}{\eta^2}$$

Note: Here, we neglect the mass term

$\Rightarrow$ For a mode k the sign flip of $\omega_k^2(\eta)$ occurs at the time:

$$\boxed{|\eta_c(k)| = \sqrt{2}/k}$$

This is also roughly the time when its proper wavelength

$$\lambda_k(\eta) = \frac{2\pi}{k} a(\eta) = \frac{2\pi}{k}\frac{1}{H|\eta|}$$

reaches the size of the Hubble horizon $d_H = 1/H$:

Check: $\lambda_k(\eta) = d_H$

is $\frac{2\pi}{k}\frac{1}{H|\eta|} = \frac{1}{H}$

$$|\eta| = \frac{2\pi}{k} \approx \frac{\sqrt{2}}{k} \quad ✓$$

The more realistic case of a de Sitter expansion of a finite duration

- Consider the case that spacetime was exponentially expanding only in a finite time interval:

$$\eta_i < \eta < \eta_f$$

and assume that spacetime was expanding slowly or was even Minkowski before η_i, and after η_f.

- **Recall:** The time when a mode, k, crosses the horizon is:

$$\eta_{hor}(k) \approx -\frac{\sqrt{2}}{k}$$

$\Longrightarrow$ modes k with $\eta_{hor}(k) \notin [\eta_i, \eta_f]$ never cross the de Sitter horizon!

$\Longrightarrow$ Three classes of modes:

1. "Small" modes:

By the time their proper wavelength would reach the Hubble horizon length the de Sitter period is already over:

$$\eta_{hor}(k) >> \eta_f$$ Recall: Both sides are negative

i.e.: $$\frac{\sqrt{2}}{k} \ll |\eta_f|$$ Recall: $\eta_{hor} \approx \frac{\sqrt{2}}{k}$

$$L \ll |\eta_f|$$

Their quantum fluctuations do not get "amplified", as we will see.

2. "Medium" size modes:

These are the modes which do cross the horizon because

$$\eta_i < \eta_{hor}(k) < \eta_f$$

The quantum fluctuations of those modes are important in cosmology.

3. "Large" modes:

These are modes which were larger than the horizon already at η_i. In the inflationary scenario they are today very much larger than the visible universe. They may only contribute, effectively, like a cosmological constant - and may even be the origin of Λ.

Quantum fluctuations in de Sitter space.

- The usual ansatz

$$\hat{\chi}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k(\eta)\, a_{-k}^+\right)$$

succeeds, as always, for any function v_k which obeys:

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0 \qquad \text{(a)}$$

$$v_k'(\eta)\, v_k^*(\eta) - v_k(\eta)\, v_k'^*(\eta) = 2i \qquad \text{(b)}$$

- The solution space of (a) can be shown to be spanned, for example, by these two real-valued Bessel functions

$$u_k(\eta) := \sqrt{k|\eta|}\; J_n(k|\eta|)$$

(not complex conjugation, just another symbol) $\rightarrow$ $$\bar{u}_k(\eta) := \sqrt{k|\eta|}\; Y_n(k|\eta|)$$

$\leftarrow$ generalizations of sine and cosine

where:

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

Thus: every mode function v_k is a linear combination

$$v_k(\eta) = A_k\, u_k(\eta) + B_k\, \bar{u}_k(\eta) \qquad (*)$$

with complex coefficients A_k, B_k.

How to identify the state of the system?

Strategy:

a.) Check if modes start out in an adiabatic regime (the small and medium ones do).

b.) Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then.

c.) Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Then: d.) Calculate $\delta\phi_k$ at the end of the exponential expansion, η_f, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3\, |v_k(\eta_f)|^2$$

Important: We know that v_k is a linear combination of u_k and $\bar{u}_k$ and we know u_k and $\bar{u}_k$ explicitly. Thus, we only need to find A_k and B_k in $(*)$!

a.) Check if modes start out in an adiabatic regime.

Indeed, we see from the K.G. eqn.

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

that at very early times, $\eta \ll 0$, we have roughly Minkowski:

$$v_k''(\eta) + k^2 v_k(\eta) = 0$$

(except if k is very small, i.e., for very large modes.)

b.) Postulate that the state $|\Omega\rangle$ of the system is the state which was the adiabatic vacuum $|vac_{early}\rangle$ then - namely the Minkowski vacuum.

Note: we could also use the adiabatic vacuum criterion, with little difference.

c.) Choose mode function v_k whose $|0\rangle$ obeys:

$$|0\rangle = |vac_{early}\rangle = |\Omega\rangle$$

Thus, v_k is the usual Minkowski mode function at early times:

$$v_k = \frac{1}{\sqrt{\omega_k}}\, e^{i\omega_k\eta + i\alpha} \quad \text{for } \eta \ll 0$$

(we are neglecting the mass term for simplicity, and because it is realistic) $\rightarrow$ i.e.

$$v_k = \frac{1}{\sqrt{k}}\, e^{ik\eta + i\alpha} \quad \text{for } \eta \ll 0$$

Technical observation: At early times, $\eta \ll 0$:

$$u_k(\eta) \approx \sqrt{\frac{2}{\pi}} \cos(k|\eta| + \text{const})$$

same constant

$$\bar{u}_k(\eta) \approx \sqrt{\frac{2}{\pi}} \sin(k|\eta| + \text{const})$$

⇒ **Proposition:**

In terms of u_k, $\bar{u}_k$ the mode function v_k reads:

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}}_{= A_k} u_k(\eta) - i \underbrace{\sqrt{\frac{\pi}{2k}}}_{= B_k} \bar{u}_k(\eta)$$

i.e.:

$$\boxed{v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}} \left(J_m(k|\eta|) - i\, Y_m(k|\eta|) \right)}$$

Proof: Exercise.

d.) Now we can calculate $\delta\phi_k$ at the end of the exponential expansion, η_f, namely:

$$\delta\phi_k(\eta_f)^2 = a^{-2}(\eta_f)\, k^3 \, |v_k(\eta_f)|^2$$

Case 1: Very small modes

- They are those with k large enough, so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

the k^2 term dominates all through the expansion.

- These modes never cross the horizon and we have, approximately:

I.e., the Bessel functions in the mode function stay sine and cosine in good approximation for all times η up to η_f.

$$v_k(\eta) = \frac{1}{\sqrt{k}} e^{ik\eta} \quad \text{for all } \eta$$

- Thus:

The vacuum fluctuations at the end of the exponential expansion are still as in Minkowski case:

Recall:

$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} |v_k(\eta_f)|$

$$\delta\phi_\lambda(\eta_f) = a^{-1}(\eta_f)\, k^{3/2} \frac{1}{\sqrt{k}}\Big|_{k \approx L^{-1}}$$

$$= \frac{1}{a(\eta_f) L}$$

$$= 1/\lambda(\eta_f)$$

proper wavelength at time η_f. (neglecting factors of 2π)

- Recall: This is the usual fluctuation spectrum for massless fields in Minkowski space:

Fluctuations with large proper spatial extent λ are still suppressed.

Case 2: Medium size modes.

- They are those with k so that in

$$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$$

the sign changes at a time $\eta_{hor}(k)$ during the exponential expansion:

$$\eta_i < \eta_{hor}(k) < \eta_f$$

Let us evaluate the fluctuation spectrum

$$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2}\, |v_k(\eta_f)|$$

at the time η_f, i.e., when the exponential expansion ends:

Then, the K.G. eqn. is to a good approximation:

Recall:

$v_k''(\eta) + \left(k^2 + \frac{m^2}{H^2\eta^2} - \frac{2}{\eta^2}\right) v_k(\eta) = 0$

$$v_k''(\eta) - \frac{2}{\eta^2}\, v_k(\eta) = 0$$

and a basis of solutions is easy to find, e.g.:

$$w_k^{(1)}(\eta) = (k|\eta|)^2 \qquad \text{decaying for } \eta \to 0$$

$$w_k^{(2)}(\eta) = \frac{1}{k|\eta|} \qquad \text{growing for } \eta \to 0$$

Recall:

$u_k(\eta) := \sqrt{k|\eta|}\, J_n(k|\eta|)$

$\bar{u}_k(\eta) := \sqrt{k|\eta|}\, Y_n(k|\eta|)$

$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{3}{2}$

Indeed: use this property of the Bessel functions:

$$\left.\begin{aligned} u_k(\eta) &\to \frac{2^{-n}}{\Gamma(n+1)}(k|\eta|)^{n+\frac{1}{2}} \to 0 \\ \bar{u}_k(\eta) &\to \frac{-\Gamma(n)}{\pi}\, 2^n\, (k|\eta|)^{\frac{1}{2}-n} \to \infty \end{aligned}\right\} \text{as } \eta \to 0 \text{ (i.e. as } t \to \infty)$$

Recall:

$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}\left(J_n(k|\eta|) - i Y_n(k|\eta|)\right)$

Therefore, for late η:

$$v_k(\eta) = \underbrace{\sqrt{\frac{\pi}{2k}}\,\frac{\Gamma(n)}{\pi}\,2^n}_{A_k}\,(k|\eta|)^{\frac{1}{2}-n} + \text{negligible}$$

Recall:

$\delta\phi_k(\eta_f) = a^{-1}(\eta_f)\, k^{3/2}\, |v_k(\eta_f)|$

$$\delta\phi_L(\eta_f) \approx \left.\underbrace{H\,\eta_f}_{a^{-1}(\eta_f)}\, k^{3/2} \sqrt{\frac{\pi}{2k}}\,\frac{\Gamma(n)}{\pi}\, 2^n (k|\eta_f|)^{\frac{1}{2}-n}\right|_{k=L^{-1}}$$

$$\Rightarrow \quad \delta\phi_L(\eta_f) \approx H\left(\frac{|\eta|}{L}\right)^{\overbrace{\frac{3}{2}-n}^{\approx 0}} \cdot \Gamma(n)\,\frac{2^n}{\pi}$$

independent of η_f! $\Rightarrow$ May as well evaluate right after horizon crossing.

For comparison, recall case 1, small modes, whose fluctuation amplitudes are as on Minkowski space: $\delta\phi_\lambda \approx 1/\lambda$

$$\boxed{\delta\phi_L(\eta_f) \approx H\cdot 2^{3/2}\,\Gamma(3/2)/\pi} \quad \text{for } n = 3/2$$

independent of $L \Rightarrow$ indep. also of λ!

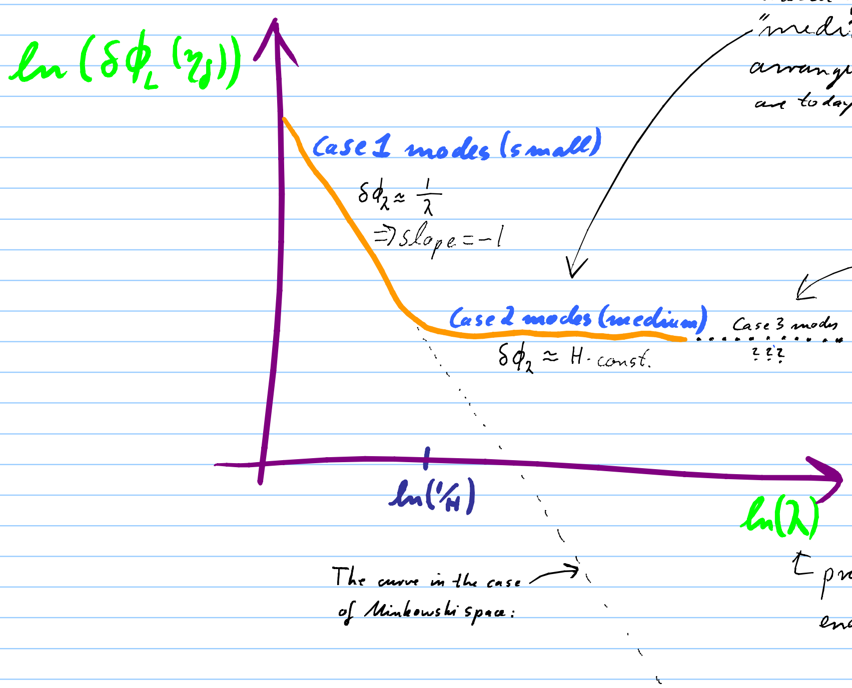
$\Rightarrow$ The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is compensated.

$\Longrightarrow$ The quantum fluctuations of a comoving mode when its proper wavelength λ is getting larger than the Hubble length, i.e., when $\lambda > \lambda_{Hubble} = 1/H$, remain as large in amplitude as they where when $\lambda = \lambda_{Hubble} = 1/H$

Indeed: $\delta\phi_L(\eta_f)$ does not depend on η_f: Fluctuations stay of same amplitude during de Sitter expansion, even though their physical wavelength grows!

$\Rightarrow$ After exponential expansion:

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are today at cosmological scales.



unknown significance (depends on assumption about their initial conditions before the expansion, at η_i: there was no vacuum state for them!)

The curve in the case of Minkowski space:

$\uparrow$ proper wavelength at η_f, the end of the exponential expansion

Preliminary estimates:

* If this is the seeding mechanism for cosmic structure formation, then:

* H determines the amplitude of the later-observed fluctuations and must be of the right size to conform with observations. Measurements of the CMB indicate:

$$H^{-1} \approx 10^5 \, l_{planck} \approx 10^{-29} m$$

* The interval $[\eta_i, \eta_f]$ must be long enough so that such small modes have time to expand to cosmological size. For example this time period would do:

$$[10^{-34} s, \ 10^{-32} s]$$

$\Rightarrow$ how much expansion?

$$\frac{a(t_f)}{a(t_i)} = e^{H(t_f - t_i)c} = e^{\frac{10^{-32} s \cdot 3 \cdot 10^8 m}{10^{-29} m \cdot s}} = e^{3 \cdot 10^5}$$