Recall de Sitter model spacetime:

\[ a(t) := e^{Ht} \]

\[ \eta(\xi) = -\frac{1}{H} e^{-H\xi} \]

\[ a(\eta) = -\frac{1}{H\eta} \]

\[ \xi''(\eta) + \omega_{x}^{2}(\eta) \xi'(\eta) = 0 \]

\[ \omega_{x}^{2}(\eta) = k^2 - \frac{2}{\eta^2} \]

Note: Here, we neglect the mass term.

⇒ For a mode \( k \) the sign flip of \( \omega_{x}^{2}(\eta) \) occurs at the time:

\[ |\xi'(k)| = \frac{72}{k} \]

This is also roughly the time when its proper wavelength

\[ \lambda_{k}(\eta) = \frac{2\pi}{k} a(\eta) = \frac{2\pi}{k} \frac{1}{H\eta} \]

reaches the size of the Hubble horizon \( d_{H} = 1/H \):

Check:

\[ \lambda_{k}(\eta) = d_{H} \]

\[ \frac{2\pi}{k} \frac{1}{H\eta} = \frac{1}{H} \]

\[ |\eta| = \frac{2\pi}{k} \approx \frac{72}{k} \quad \checkmark \]
The more realistic case of a de Sitter expansion of a finite duration

Consider the case that spacetime was exponentially expanding only in a finite time interval:

\[ \eta_i < \eta < \eta_f \]

and assume that spacetime was expanding slowly or was even Minkowski before \( \eta_i \), and after \( \eta_f \).

Recall: The time when a mode, \( k \), crosses the horizon is:

\[ \eta_{\text{hor}}(k) \approx -\frac{\eta_f}{k} \]

\[ \Rightarrow \] modes \( k \) with \( \eta_{\text{hor}}(k) \notin [\eta_f, \eta_i] \) never cross the de Sitter horizon!

\[ \Rightarrow \] Three classes of modes:

1. "Small" modes:

By the time their proper wavelength would reach the Hubble horizon length the de Sitter period is already over:

\[ \eta_{\text{hor}}(k) \gg \eta_f \] \hspace{1cm} \text{Recall: Both sides are negative}

\[ \frac{\eta_f}{k} \ll 1 \eta_f \] \hspace{1cm} \text{Recall: } \eta_{\text{hor}} \approx -\frac{\eta_f}{k}

\[ L \ll 1 \eta_f \]

Their quantum fluctuations do not get "amplified", as we will see.
2. "Medium" size modes:

These are the modes which do cross the horizon because

$$\eta_i < \eta_{hm} (k) < \eta_f$$

The quantum fluctuations of these modes are important in cosmology.

3. "Large" modes:

These are modes which were longer than the horizon already at $\eta_i$. In the inflationary scenario they are today very much longer than the visible universe. They may only contribute effectively, like a cosmological constant, and may even be the origin of $\Lambda$.

Quantum fluctuations in de Sitter space.

The usual ansatz

$$\hat{\mathcal{F}}_n (\eta) = \frac{1}{\sqrt{2}} \left( V_n^* (\eta) \alpha_k + V_n (\eta) \beta_k^* \right)$$

succeeds, as always, for any function $V_n$ which obeys:

$$V_n '' (\eta) + \left( k^2 + \frac{m^2}{\eta^2} - \frac{2}{\eta^2} \right) V_n (\eta) = 0 \quad (a)$$

$$V_n ' (\eta) V_n ' (\eta) - V_n (\eta) V_n '' (\eta) = 2i \quad (b)$$

The solution space of (a) can be shown to be spanned, for example, by these two real-valued Bessel functions...
\[ u_k(\eta) := \sqrt{k! \eta!} J_n(k \eta) \]

(not complex conjugation) \[ \overline{u}_k(\eta) := \sqrt{k! \eta!} Y_n(k \eta) \] in general extensions of sine and cosine

where:

\[ n = \sqrt{\frac{q}{4} - \frac{m^2}{h^2}} \]

Thus: every mode function \( v_k \) is a linear combination

\[ v_k(\eta) = A_k u_k(\eta) + B_k \overline{u}_k(\eta) \] (X)

with complex coefficients \( A_k, B_k \).

How to identify the state of the system?

Strategy:

a.) Check if modes start out in an adiabatic regime (the small and medium ones do).

b.) Postulate that the state of the system is the state which was the adiabatic vacuum \( \text{vac}_{\text{early}} \) then.

c.) Choose mode function \( v_k \) whose \( \Omega \) obeys:

\[ \Omega = \langle \text{vac}_{\text{early}} | \Omega \rangle \]

Then: d.) Calculate \( S \phi \) at the end of the exponential expansion, \( \Omega \), namely:
\[ \delta \Phi_k (\eta) = \alpha^2 (\eta) k^3 \left| \nu_k (\eta) \right|^2 \]

Important: We know that \( \nu_k \) is a linear combination of \( u_k \) and \( \overline{u}_k \) and we know \( u_k \) and \( \overline{u}_k \) explicitly. Thus, we only need to find \( A_k \) and \( B_k \) in (6).

\[ a) \text{ Check if modes start out in an adiabatic regime.} \]

Indeed, we see from the K.C. eqn.

\[ v_k''(\eta) + \left( k^2 + \frac{m^2}{\hbar^2 \eta^2} - \frac{2}{\eta^2} \right) v_k(\eta) = 0 \]

that at very early times, \( \eta \ll 0 \), we have roughly Minkowski:

\[ v_k''(\eta) + k^2 v_k(\eta) = 0 \quad (\text{except if } k \text{ is very small,}) \]

i.e., for very large modes.

\[ b) \text{ Postulate that the state } |\Omega> \text{ of the system is in the state which was the adiabatic vacuum } |\nu_{\text{early}}> \text{ there}. \]

- namely, the Minkowski vacuum.

Note: we could also use the adiabatic vacuum criterion, with little difference.

\[ c) \text{ Choose mode function } \nu_k \text{ whose } |\Omega> \text{ obeys:} \]

\[ |\Omega> = |\nu_{\text{early}}> = |\nu_k> \]

Thus, \( \nu_k \) is the usual Minkowski mode function at early times:

\[ \nu_k = \frac{1}{\sqrt{\nu_k}} e^{i \omega_k \xi} \quad \text{for } \eta < 0 \]

\[ \text{we are neglecting the mass term for simplicity, and because it is negligible} \]

\[ \rightarrow \text{ i.e., } \nu_k = \frac{1}{\sqrt{\nu_k}} e^{i \xi \xi} \quad \text{for } \eta < 0 \]
Technical Observation: At early times, \( \eta \ll \epsilon \):

\[
\begin{align*}
\varphi_k(\eta) &\approx \frac{\sqrt{2}}{\pi} \cos(k \eta + \text{const}) \\
\bar{\varphi}_k(\eta) &\approx \frac{\sqrt{2}}{\pi} \sin(k \eta + \text{const})
\end{align*}
\]

\( \Rightarrow \) Proposition:

In terms of \( \varphi_k \), \( \bar{\varphi}_k \) the mode function \( \nu_k \) needs:

\[
\nu_k(\eta) = \frac{\sqrt{\pi \sigma}}{2k} \varphi_k(\eta) - i \frac{\sqrt{\pi \sigma}}{2k} \bar{\varphi}_k(\eta)
\]

i.e.:

\[
\nu_k(\eta) = \frac{\sqrt{\pi \sigma}}{2} \left( J_m(k \eta) - i Y_m(k \eta) \right)
\]

Proof: Exercise.

d) Now we can calculate \( S \Phi_k \) at the end of the exponential expansion, \( \eta_f \), namely:

\[
S \Phi_k(\eta_f)^2 = a^2(\eta_f) k^3 |\nu_k(\eta_f)|^2
\]

Case 1: Very small modes

They are those with \( k \) large enough, so that in

\[
\nu_k''(\eta) + \left( k^2 + \frac{m^2}{\lambda^2 \eta^2} - \frac{2}{\gamma^2} \right) \nu_k(\eta) = 0
\]

the \( k^2 \) term dominates all through the expansion.
These modes never cross the horizon and we
have, approximately:

\[ V_k (\eta) = \frac{i}{\sqrt{8\pi}} e^{i k \eta} \quad \text{for all } \eta \]

Thus:

The vacuum fluctuations at the end
of the exponential expansion are still
as in Minkowski case:

Recall:

\[ \delta \phi_k (\eta) = a^i (\eta) k^{3/2} \frac{1}{V_k} \left| \eta \right| \]

= \frac{1}{\lambda (\eta)}

= \frac{1}{\lambda (\eta)}

(proper wavelength
at time \eta)

(neglecting factors of 2π)

Recall: This is the usual fluctuation spectrum
for massless fields in Minkowski space:

Fluctuations with large proper
spatial extent \lambda are still suppressed.

Case 2: Medium size modes.

They are those with \( k \) so that in

\[ V_k (\eta) + \left( k^2 + \frac{m^2}{H^2 \eta^2} - \frac{2}{\eta^2} \right) V_k (\eta) = 0 \]

the sign changes at a time \( \eta_o (k) \) during the exponential expansion:

\[ \eta_i < \eta_o (k) < \eta_f \]
Let us evaluate the fluctuation spectrum

$$\delta \Phi_n (\eta) = a^{n-1}(\eta) k^{3/2} |v_n (\eta)|$$

at the time $\eta_f$, i.e., when the exponential expansion ends:

Then, the K.B. eqn. is to a good approximation:

Recall:

$$\nu''(\eta) + \left( k^2 + \frac{\omega^2}{\mu_\eta^2} - \frac{2}{\eta^2} \right) \nu_n(\eta) = 0$$

$$\nu''_k(\eta) - \frac{2}{\eta^2} \nu_k(\eta) = 0$$

and a basis of solutions is easy to find, e.g.:

$$\nu_k^{(1)}(\eta) = (k1\eta_1)^2 \quad \text{decaying for } \eta \to 0$$

$$\nu_k^{(0)}(\eta) = \frac{1}{k1\eta_1} \quad \text{growing for } \eta \to 0$$

Indeed: use this property of the Bessel functions:

Recall:

$$u_n (\eta) = \sqrt{\frac{n}{\mu_\eta}} J_n (k1\eta_1)$$

$$\tilde{u}_n (\eta) = \sqrt{\frac{n}{\mu_\eta}} Y_n (k1\eta_1)$$

$$m = \sqrt{\frac{\sigma}{\mu_\eta}} \approx \frac{3}{2}$$

$$\tilde{u}_n (\eta) \to -\frac{\Gamma(n)}{\pi} 2^{-m} (k1\eta_1)^{1/2-m} \to 0 \quad \text{as } \eta \to 0$$

Recall:

Therefore, for late $\eta$:

$$v_k(\eta) = \sqrt{\frac{\pi}{2k}} \frac{\Gamma(n)}{\pi} 2^{-m} (k1\eta_1)^{1/2-m} + \text{negligible}$$

Recall:

$$\delta \Phi_k (\eta) = a^{n-1}(\eta) k^{3/2} |v_k (\eta)|$$

$$\delta \Phi_k (\eta) \approx H \eta_k k^{3/2} \sqrt{\frac{\pi}{2k}} \frac{\Gamma(n)}{\pi} 2^{-m} (k1\eta_1)^{1/2-m}$$
\[ \Rightarrow \delta \phi_c(\eta_0) \approx H(\eta_0) \frac{3}{2} \frac{n}{n-2} \cdot \frac{a(\eta)}{\eta} \approx \text{independent of } \eta_0 \] 

\[ \delta \phi_c(\eta) \approx H \cdot 2^{3/2} a(\eta)/\pi \quad \text{for } n = 3/2 \]

\[ \Rightarrow \text{The medium sized modes get amplified just enough so that the usual suppression of fluctuations of large spatial extent is compensated.} \]

\[ \Rightarrow \text{The quantum fluctuations of a comoving mode, when its proper wavelength } \lambda \text{ is getting longer than the Hubble length, i.e., when } \lambda > \lambda_{\text{Hubble}} = \frac{1}{H}, \text{ remain as large in amplitude as they were when } \lambda = \lambda_{\text{Hubble}} = \frac{1}{H} \]

\[ \text{Indeed: } \delta \phi_c(\eta) \text{ does not depend on } \eta_0: \text{ Fluctuations stay of same amplitude during de Sitter expansion.} \]

\[ \Rightarrow \text{After exponential expansion:} \]

\[ \ln(\delta \phi_c(\eta)) \]

- Case 1 modes (small):
  \[ \delta \phi_c \approx \frac{1}{\lambda} \Rightarrow \text{slope} = -1 \]

- Case 2 modes (medium):
  \[ \delta \phi_c \approx H \cdot \text{const.} \]

- Case 3 modes (large):
  \[ \delta \phi_c \approx \text{constant} \]

As we'll see, in a suitable model of very early universe cosmology, "medium size" can be arranged to mean modes that are tiny at cosmological scales.

[Diagram: Log-log plot of \( \ln(\delta \phi_c(\eta)) \) vs. \( \ln(\lambda) \).]

The same as the case proper wavelength at \( \eta_0 \), the end of the exponential expansion.
Preliminary estimates:

* If this is the seeding mechanism for cosmic structure formation, then:

* $H$ determines the amplitude of the late-observed fluctuations and must be of the right size to conform with observations. Measurements of the CMB indicate:

$$H^2 \approx 10^{-5} \text{m/s}^2 \approx 10^{-29} \text{m}$$

* The interval $[\tau = \frac{1}{2}]$ must be long enough so that such small modes have time to expand to cosmological size. For example, this time period would do:

$$[10^{-44} \, \text{s}, 10^{-32} \, \text{s}]$$