QFT for Cosmology, Achim Kempf, Winter 16, Lecture 23

Note Title

Plan: Unruh effect through Bogolubov transformations

$\Downarrow$

Hawking effect

Unruh effect in 1+1 dimensions

(Metric convention: $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)

Consider an observer's trajectory $x^\alpha(\tau)$ and use the observer's proper time τ as the parameter.

- **Velocity** $\dot{x}^\alpha(\tau) := \frac{dx^\alpha(\tau)}{d\tau}$

Proposition: $\eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1 \quad \forall \tau$

Proof: At any point in time, τ, in rest frame: $x^\mu(\tau) = (\tau, 0)$

$\Rightarrow \dot{x}^\mu(\tau) = (1, 0)$

$\Rightarrow \eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1$ which is a scalar:

$\Rightarrow \eta_{\mu\nu}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) = 1$ in all cd systems for all τ.

- **Acceleration:** $\ddot{x}^\alpha(\tau) := \frac{d\dot{x}^\alpha(\tau)}{d\tau}$

Proposition: $\ddot{x}_\mu(\tau)\dot{x}^\mu(\tau) = 0 \quad \forall \tau$

Proof: $0 = \frac{d}{d\tau}\left(\dot{x}_\mu(\tau)\dot{x}^\mu(\tau)\right) = 2\ddot{x}_\mu(\tau)\dot{x}^\mu(\tau)$

"proper acceleration"

$a(\tau) := \frac{d^2x^1(\tau)}{d\tau^2}$

$\Rightarrow$ In rest frame: $\dot{x}^\mu(\tau) = (1, 0)$ and $\ddot{x}^\mu(\tau) = (0, a(t))$

$\Rightarrow$ In every frame: $\ddot{x}_\mu(\tau)\ddot{x}^\mu(\tau) = -a^2(\tau)$

Special case of uniform acceleration: $a(\tau) = a \quad \forall \tau$

Proposition: A trajectory of uniform acceleration a is given by:

$$x^\mu(\tau) = \left(t(\tau), x(\tau)\right) = \left(\tfrac{1}{a}\sinh(a\tau), \tfrac{1}{a}\cosh(a\tau)\right)$$

Proof: $\dot{x}_\mu(\tau) = \left(\cosh(a\tau), \sinh(a\tau)\right)$

is obeying $\dot{x}_\mu \dot{x}^\mu = 1$ ✓

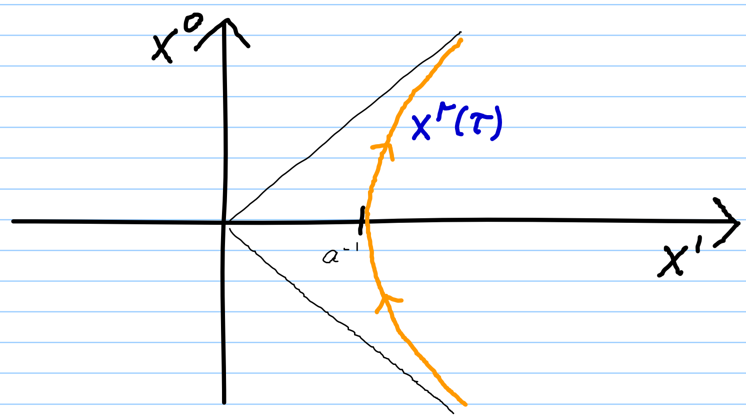
$\Rightarrow$ τ really is the proper time

And, crucially:

$\ddot{x}_\mu(\tau) = \left(a\sinh(a\tau), a\cosh(a\tau)\right)$ obeying $\dot{x}_\mu \ddot{x}^\mu = -a^2$ ✓

This trajectory also obeys:

$$x_\mu(\tau)x^\mu(\tau) = x^0(\tau)^2 - x^1(\tau)^2 = -\frac{1}{a^2}$$



i.e., it is a hyperbola of deceleration followed by acceleration.

Notice: Our uniformly accelerated traveler has horizons:

- can't influence events below the line $x^0 = -x^1$, i.e., with $x^0 + x^1 \leq 0$
- can't be influenced by events above the line $x^0 = x^1$, i.e., with $x^0 - x^1 \geq 0$

(A)

Inertial light cone coordinate system:

The inertial cartesian coordinates are fine to describe particle motion.

For wave equations, often light cone coordinates have advantages. (Esp. in 1+1D):

$$\tilde{x}^\mu(x^0,x^1) := \left(u(x^0,x^1),\, v(x^0,x^1)\right)$$

with:
$$\left.\begin{aligned} u(x^0,x^1) &:= x^0 - x^1 \\ v(x^0,x^1) &:= x^0 + x^1 \end{aligned}\right\} (B)$$

The metric: In inertial, cartesian cds x^μ: $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix}$

$$\boxed{\text{In inertial light cone cds } \tilde{x}^\mu: \quad g_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0, 1/2 \\ 1/2, 0 \end{pmatrix} \quad \text{i.e.: } ds^2 = du\,dv}$$

Exercise: Check this, using $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x)$.

The trajectory above in inertial light cone cds:

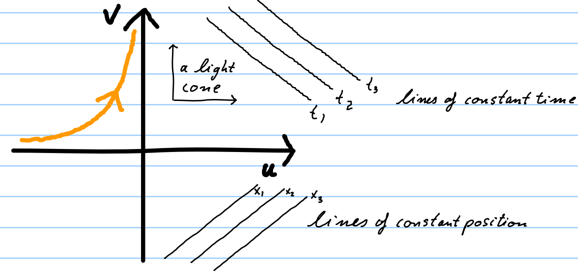
$$\tilde{x}(\tau) = (u(\tau), v(\tau))$$

with $u(\tau) = t(\tau) - x(\tau) = -\frac{1}{a} e^{-a\tau}$

$v(\tau) = t(\tau) + x(\tau) = \frac{1}{a} e^{a\tau}$

Notice: From (A) $\wedge$ (B): the traveller

- can't influence events (u,v) with $v \leq 0$
- can't be influenced by events (u,v) with $u \geq 0$



A coordinate system that is comoving with the traveler

We want a coordinate system ξ^μ so that our traveler's trajectory is:

$$\xi^\mu(\tau) = (\tau, 0)$$

But this fixes the new cds only on the trajectory!

Q: How to continue the new cds to elsewhere?

A: We can require (in 1+1 dimensions) that the light cones are still at 45°, i.e., that

$$g_{\mu\nu}(\xi) = f(\xi)\begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix},$$

i.e. $ds^2 = f(\xi)\left(d\xi^{0\,2} - d\xi^{1\,2}\right)$, i.e., $\underbrace{ds^2 = 0}_{\substack{\text{condition for} \\ \text{light-like tangent.}}} \Rightarrow d\xi^1 = \pm d\xi^0$

Proposition:

Under the change of coordinates

$$\left.\begin{aligned} x^0(\xi) &= a^{-1} e^{a\xi^1} \sinh(a\xi^0) \\ x^1(\xi) &= a^{-1} e^{a\xi^1} \cosh(a\xi^0) \end{aligned}\right\} (T)$$

we have that the trajectory $\xi^\mu(\tau) = (\tau, 0)$ is indeed the trajectory of our traveler:

$$x^\mu(\tau) = \left(a^{-1}\sinh(a\tau), a^{-1}\cosh(a\tau)\right)$$

And in addition: The Minkowski metric $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reads in the ξ coordinates:

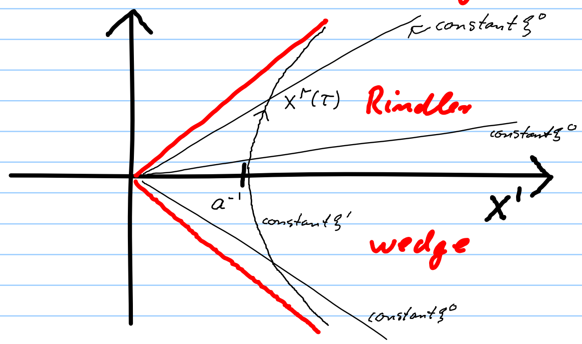
$$\boxed{g_{\alpha\beta}(\xi) = e^{2a\xi^1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$\Rightarrow$ In this cds, light travels still at 45°.

In (T), why did we map the new cds to the old: $\xi^\mu \to x^\alpha$?

There is no inverse $x^\alpha \to \xi^\mu$!

Why? Because all of $(\xi^0, \xi^1) \in \mathbb{R}^2$ maps only on to the Rindler wedge $x^1 > |x^0|$



From (T):

For each ξ^1, obtain a hyperbola within the Rindler wedge.

Together they cover exactly only the Rindler wedge.

We knew that the traveler has horizons.

His comoving cds ξ^μ reaches only as far as to his horizons.

Accelerated light cone coordinates.

In ξ^μ cds, light still travels at 45°.

$\Rightarrow$ It will be useful for wave equations to introduce accelerated light cone coordinates:

$$\tilde{\xi}^\mu(\xi) = \left(\tilde{\xi}^0(\xi), \tilde{\xi}^1(\xi)\right) = \left(\bar{u}(\xi), \bar{v}(\xi)\right)$$

where:
$$\bar{u}(\xi) = \xi^0 - \xi^1$$
$$\bar{v}(\xi) = \xi^0 + \xi^1$$

In the cds $\tilde{\xi}^\mu$ we have:

$$\boxed{\begin{aligned} g_{\mu\nu}(\tilde{\xi}) &= e^{a(\bar{v}-\bar{u})}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \\ \text{i.e.: } ds^2 &= e^{a(\bar{v}-\bar{u})}\, d\bar{u}\, d\bar{v} \end{aligned}}$$

Remark: We can also directly map the accelerated light cone cds $\tilde{\bar\xi} = (\bar u, \bar v)$ into the inertial light cone coordinates $\tilde{X} = (u, v)$: (Exercise: show this)

Important later! → $\boxed{u(\bar u, \bar v) = -\frac{1}{a} e^{-a\bar u}}$ $\quad v(\bar u, \bar v) = \frac{1}{a} e^{a\bar v}$

Summary:

Coordinate system	Form of the metric
$X = (x^0, x^1)$	$g_{\mu\nu}(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mu\nu}$
$\tilde X = (u, v)$	$g_{\mu\nu}(\tilde x) = \begin{pmatrix} 0 & \frac12 \\ \frac12 & 0 \end{pmatrix}$
$\xi = (\xi^0, \xi^1)$	$g_{\mu\nu}(\xi) = e^{2a\xi^1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\bar\xi = (\bar u, \bar v)$	$g_{\mu\nu}(\bar\xi) = e^{a(\bar v - \bar u)} \begin{pmatrix} 0 & \frac12 \\ \frac12 & 0 \end{pmatrix}$

These cds cover only the Rindler wedge (the last two coordinate systems, ξ and $\bar\xi$)

Observation: These metrics are pairwise conformally related:

$$g_{\mu\nu}(x) \longrightarrow \bar g_{\mu\nu}(x) := \Omega^2(x)\, g_{\mu\nu}(x)$$

In 2 dimensions, the K.G. action is invariant:

$$S_g[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{|g|}\,d^2x$$

$$= S_{\bar g}[\phi]$$

Proof:

We have $g^{\mu\nu}(x) \to \bar g^{\mu\nu}(x) = \Omega^{-2}(x)\, g^{\mu\nu}(x)$

and $\sqrt{|g|} \to \sqrt{|\bar g|} = \Omega^2(x)\sqrt{|g|}$ in 2 dimensions.

⇒ The Klein Gordon action

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\,\sqrt{|g|}\,d^2x \qquad \text{general cds}$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}(\partial_{x^0}\phi)^2-(\partial_{x^1}\phi)^2\,dx^0dx^1 \qquad \text{inertial cartesian cds}$$

$$= 2\int_{\mathbb{R}^2}(\partial_u\phi)(\partial_v\phi)\,du\,dv \qquad \text{inertial light cone cds}$$

On Rindler Wedge: (easy to see because of conformal invariance)

$$S_{RW}[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}(\partial_{\xi^0}\phi)^2-(\partial_{\xi^1}\phi)^2\,d\xi^0d\xi^1 \qquad \text{accelerated cartesian cds}$$

$$= 2\int_{\mathbb{R}^2}(\partial_{\bar u}\phi)(\partial_{\bar v}\phi)\,du\,dv \qquad \text{accelerated light cone cds}$$

because massive action is not conformal

Remark: A massive field would have a different equation motion in accelerated frames.

i.e.: accelerated observer can find out he's accelerating using masses.

The Klein Gordon equation:

In inertial light cone coordinates:

$$\frac{\delta S}{\delta\phi} = \partial_u\frac{\delta S}{\delta\partial_u\phi}+\partial_v\frac{\delta S}{\delta\partial_v\phi} \qquad \text{i.e.}\quad \partial_u\partial_v\phi(u,v)=0$$

Easily solved: $\phi(u,v) = A(u) + B(v)$, with A, B arbitrary functions.

For example: $\phi(u,v) = e^{-i\omega u} = e^{-i\omega(t-x)} = e^{i\omega(x^0-x^1)}$

is a right-moving positive frequency solution.

The usual Minkowski space quantum field solution $\hat\phi(x^0,x^1)$ can be written this way:

$$\hat\phi(u,v) = \int_0^\infty \frac{dk}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\Big(\underbrace{e^{-i\omega u}\hat a_k + e^{i\omega u}\hat a_k^+}_{\text{right movers}} + \underbrace{e^{-i\omega v}\hat a_{-k} + e^{i\omega v}\hat a_{-k}^+}_{\text{left movers}}\Big) \quad \text{and } \omega=|k|$$

The Klein Gordon equation in the accelerated frame:

In accelerated light cone coordinates: (covering only the Rindler wedge)

$$\frac{\delta S'_{RW}}{\delta \phi} = \partial_{\bar{u}} \frac{\delta S'_{RW}}{\delta \partial_{\bar{u}}\phi} + \partial_{\bar{v}} \frac{\delta S'_{RW}}{\delta \partial_{\bar{v}}\phi} \quad \text{i.e.} \quad \partial_{\bar{u}}\partial_{\bar{v}}\phi(\bar{u},\bar{v}) = 0$$

Easily solved: $\phi(\bar{u},\bar{v}) = A(\bar{u}) + B(\bar{v})$, with A, B arbitrary functions.

For example: $\phi(\bar{u},\bar{v}) = e^{-i\omega\bar{u}} = e^{-i\omega(\xi^0-\xi^1)}$

is a right-moving positive frequency solution.

In the accelerated frame, the quantum field in the Rindler wedge is:

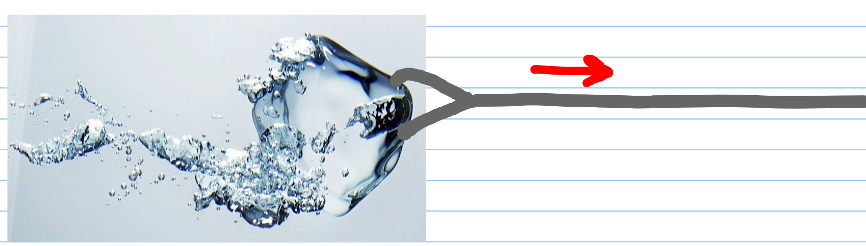
$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \Big(\underbrace{e^{-i\omega\bar{u}}\hat{b}_k + e^{i\omega\bar{u}}\hat{b}_k^+}_{\text{right movers}} + \underbrace{e^{-i\omega\bar{v}}\hat{b}_{-k} + e^{i\omega\bar{v}}\hat{b}_{-k}^+}_{\text{left movers}} \Big) \quad \text{and } \omega = |k|$$

Notice: hermiticity conditions, K.G. eqn and CCRs obeyed.

For the inertial observer, the vacuum state obeys: $a_k|0_M\rangle = 0$

But for the accelerated observer, the vacuum state obeys: $b_k|0_R\rangle = 0$

We will assume that the state of the system is $|\psi\rangle = |0_M\rangle$.



Will acceleration melt ice?

We arrived at a typical situation:

$$\hat\phi(u,v) = \int_0^\infty \frac{dk}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega u}\hat a_k + e^{i\omega u}\hat a_k^+ + e^{-i\omega v}\hat a_{-k} + e^{i\omega v}\hat a_{-k}^+\right)$$
$$= \int_0^\infty \frac{dk}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega \bar u}\hat b_k + e^{i\omega \bar u}\hat b_k^+ + e^{-i\omega \bar v}\hat b_{-k} + e^{i\omega \bar v}\hat b_{-k}^+\right) \qquad \text{(A)}$$

There must exist a Bogolubov transformation linking the a_k, a_k^+ and b_k, b_k^+!

Observation:

The left and right movers won't mix.

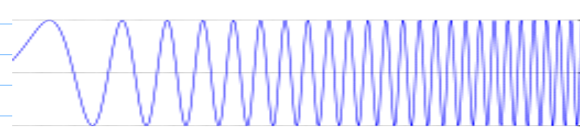
$\rightsquigarrow$ For simplicity we'll consider only the right movers.

Observation:

Among right movers all frequencies may mix:

$$b_\Omega = \int_0^\infty d\omega\,\left(\alpha_{\Omega\omega}\, a_\omega - \beta_{\Omega\omega}\, a^+_\omega\right) \quad \text{with } \omega = k \qquad \text{(B)}$$

Intuition: To the traveller, any monochromatic wave sounds like a chirp.



Exercise: Check that $[a_k, a_{k'}^+] = \delta(k-k')$ and $[b_k, b_{k'}^+] = \delta(k-k')$ imply:

$$\int_0^\infty d\omega\,\left(\alpha_{\Omega\omega}\,\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\,\beta^*_{\Omega'\omega}\right) = \delta(\Omega-\Omega') \qquad \text{(C)}$$

Calculation of $\alpha_{\Omega\omega}$ and $\beta_{\Omega\omega}$:

(lengthy, for more details, see Mukhanov & Winitzki text.)

- Substitute (B) into (A) and collect coefficients of a_ω

$$\Rightarrow \quad \omega^{-1/2}\, e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}} \left(\alpha_{\Omega'\omega}\, e^{-i\Omega'\bar{u}} - \beta^*_{\Omega'\omega}\, e^{i\Omega'\bar{u}} \right)$$

- Act with $\int_{-\infty}^{\infty} d\bar{u}\, e^{\pm i\Omega\bar{u}} \cdot$ on the equation

and then use that $\int_{-\infty}^{\infty} e^{i(\Omega-\Omega')\bar{u}}\, d\bar{u} = 2\pi\,\delta(\Omega-\Omega')$.

$\Rightarrow$

$$\begin{matrix} +\text{ case:} \\ -\text{ case:} \end{matrix} \quad \left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{\mp i\omega u + i\Omega\bar{u}}\, d\bar{u}$$

Recall: $\boxed{u(\bar{u},\bar{v}) = -\frac{1}{a}\, e^{-a\bar{u}}}$ (encoding the chirping)

and, therefore: $\frac{du}{d\bar{u}} = e^{-a\bar{u}} \Rightarrow d\bar{u} = (-au)\,du$

$\Rightarrow$

$$\left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} e^{\mp i\omega u + i\Omega\bar{u}}\, d\bar{u} = \pm \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{0} e^{\mp i\omega u}\, (-au)^{-i\frac{\Omega}{a}-1}\, du$$

- Now, using $\Gamma(r) = \int_0^\infty s^{r-1} e^{-s}\, ds$

$$\Rightarrow \quad \boxed{\left.\begin{matrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{matrix}\right\} = \pm \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\; e^{\pm\frac{\pi\Omega}{2a}}\; e^{i\left(\frac{\Omega}{a}\ln\frac{\omega}{a}\right)}\, \Gamma\!\left(-\frac{i\Omega}{a}\right)} \quad !$$

Observation: $\Rightarrow \quad |\beta_{\Omega\omega}|^2 = e^{-\frac{2\pi\Omega}{a}} |\alpha_{\Omega\omega}|^2 \qquad$ (D)

So for acceleration $a \to 0$ we have $|\beta_{\Omega\omega}| \to 0$, i.e. then no particles observed in travelers frame.

How many particles does an accelerated observer see if $a \neq 0$?

$$\langle \psi_{in} | \hat{N}_\Omega | \psi_{in} \rangle = \langle 0_M | \hat{N}_\Omega | 0_M \rangle$$

$$= \langle 0_M | \hat{b}_\Omega^+ \hat{b}_\Omega | 0_M \rangle$$

$$= \langle 0_M | \left(\int_0^\infty \alpha^*_{\omega\Omega} \hat{a}^+_\omega - \beta^*_{\omega\Omega} \hat{a}_\omega \, d\omega \right) \left(\int_0^\infty \alpha_{\omega'\Omega} \hat{a}_{\omega'} - \beta_{\omega'\Omega} \hat{a}^+_{\omega'} \, d\omega' \right) | 0_M \rangle$$

$$= \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2$$

Using (C) and (D) $\Rightarrow$

$$\langle \psi_{in} | \hat{N}_\Omega | \psi_{in} \rangle = \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \underbrace{\delta(\Omega - \Omega)}_{\uparrow \text{Divergent}}$$

Observation:

With infrared cutoff through (accelerating) box of size V we have discrete k, discrete $\Omega(k)$ and $\delta(\Omega - \Omega')$ in (C) becomes $V \delta_{\Omega,\Omega'}$.

Then:

$$\langle \psi_{in} | \hat{N}_\Omega | \psi_{in} \rangle = \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} V \underbrace{\delta_{\Omega,\Omega}}_{=1}$$

$\Rightarrow$ Number density:

$$\boxed{\bar{n}_\Omega := \frac{1}{V}\langle \psi_{in}|\hat{N}_\Omega|\psi_{in}\rangle = \frac{1}{e^{\frac{2\pi\Omega}{a}}-1}}$$

Compare: If a harmonic oscillator of energy levels $E_n = \Omega(n+\frac{1}{2})$ is exposed to a heat bath of temperature T, then its expected excitation number is

$$\bar{n} = \frac{1}{e^{\Omega/T}-1}$$

$\Rightarrow$ The traveler's mode oscillators are excited as if exposed to a heat bath of the Unruh temperature:

$$\boxed{T = \frac{a}{2\pi}}$$

Observation:

- Could the quantum field be in the state $|0_R\rangle$?
- We'd expect that then inertial observers would see particles!
- But $|0_R\rangle$ is not a physically implementable state, even in principle! Why?
- $|0_R\rangle$ is a state with regions of diverging energy density!

Why? If $\hat{\phi}$ is in state $|0_R\rangle$ then, in accelerated cds, energy density is constant throughout that cds.

But this cds piles up at the horizons!

Recall: $T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \eta_{\mu\nu}(\partial_\alpha\phi)(\partial^\alpha\phi)$

$\Rightarrow$ Need to study terms of the form $\langle 0_R|(\partial\phi)^2|0_R\rangle$.

Calculate: $\underbrace{\langle 0_R|(\partial_u\hat{\phi})^2|0_R\rangle}_{\substack{\text{enters } \langle 0_R|T_{\mu\nu}(u,v)|0_R\rangle \\ \text{calculation of} \\ \text{inertial observer!}}} = \langle 0_R|\left(\frac{\partial\bar{u}}{\partial u}\right)^2(\partial_{\bar{u}}\hat{\phi})^2|0_R\rangle$

Recall: $\boxed{u(\bar{u},\bar{v}) = -\frac{1}{a}e^{-a\bar{u}}}$

$\Rightarrow \quad \frac{du}{d\bar{u}} = -au \quad \Rightarrow$

$= \frac{1}{(au)^2}\langle 0_R|(\partial_{\bar{u}}\hat{\phi})^2|0_R\rangle$

same b/c calculated exact same way from (A)

$= \frac{1}{(au)^2}\underbrace{\langle 0_M|(\partial_u\hat{\phi})^2|0_M\rangle}_{\text{Finite after renormalization.}}$

But: $u^{-1}\to\infty$ at the traveler's horizon!

$\Rightarrow$ In states $|\psi\rangle = |0_R\rangle$, or $|\psi\rangle = b_\Omega^+|0_R\rangle$ etc,

$\langle\psi|T_{\mu\nu}(u,v)|\psi\rangle \to \infty$ as $u\to 0$ (future horizon)

and similarly also for $v\to 0$.