QFT for Cosmology, Achim Kempf, Winter 16, Lecture 24

Note Title

The Hawking effect

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity: $r = 0$

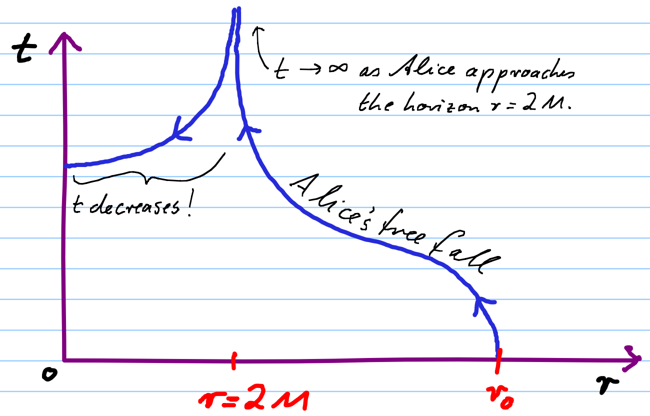
Horizon: $r = 2M$

Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

- Due to the sign changes across $r = 2M$, for $r < 2M$ dt is spacelike and dr is timelike for $r < 2M$.
- Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:



$$r(\alpha) = \frac{r_0}{2}\left(1 + \cos(\alpha)\right)$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right) w\alpha + \frac{r_0}{2} w \sin(\alpha) + 2M \log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}\left(\alpha + \sin(\alpha)\right)$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

- For quantization, need better choices of coordinate systems!

Simplification: For now, we drop the φ and θ coordinates.

First design of a new cds (T,R) - Alice's choice (for $r_0 = 2M$):

- ☐ Require $g_{\mu\nu}(T,R)$ to be regular across $r = 2M$.
- ☐ Require $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ at $r=2M$. If there's really no singularity at $r=2M$ this must be possible.
- ☐ Extend this cds so that $g_{\mu\nu}(T,R) = f(T,R)\,\eta_{\mu\nu}$ because then we know:
  - ☐ the action
  - ☐ the Klein Gordon equation
  - ☐ the solution space of the K.G. equation.
  - ☐ which is the mode fctn of the vacuum in this cds.

⇒ **Alice's choice are the Kruskal-Szekeres coordinates (T,R):**

$$T(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M)+\cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

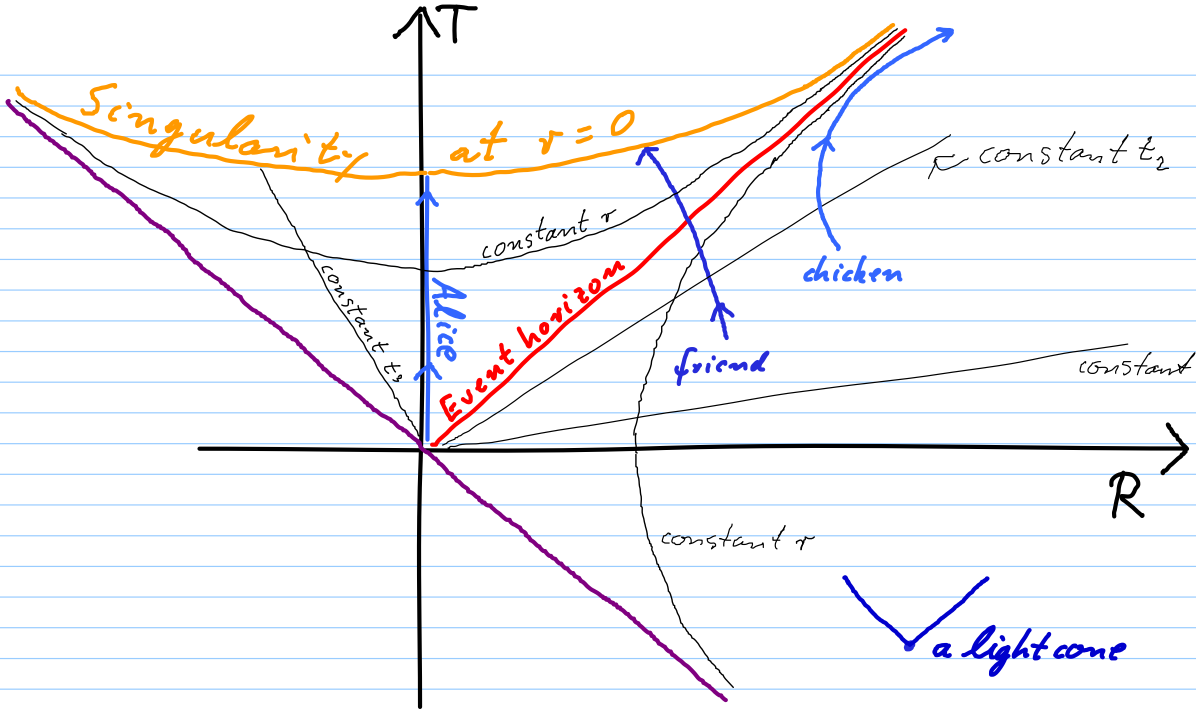
$$R(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M)+\sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}}_{\text{Conformal prefactor} = 1 \text{ as } r=2M} \underbrace{\left(dT^2 - dR^2\right)}_{\overset{\shortparallel}{\eta_{\mu\nu}}}$$

Obeys all conditions!



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor} \\ \text{(which is 1 at horizon)}}} \underbrace{du\,dv}_{\substack{\text{light cone} \\ \text{Minkowski}}}$$

⇒ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{g}\,d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T \phi(T,R)\right)^2 - \left(\partial_R \phi(T,R)\right)^2 dT\,dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u \phi(u,v)\right)\left(\partial_v \phi(u,v)\right) dv\,du$$

$0 \leftarrow$ b/c region $T > -R$ means $T + R > 0$, i.e. $v > 0$.

⇒ Eqn of motion: $\partial_u \partial_v \phi(u,v) = 0$

⇒ Solution for the QFT found as before:

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{(2\omega)^{1/2}} \left(e^{-i\omega u} \hat{a}_\omega + e^{i\omega u} \hat{a}^+_\omega + \text{left movers} \right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Alice's notion of vacuum

- ☐ For Alice, as she crosses the horizon, $g_{\mu\nu} = \eta_{\mu\nu}$.
- ☐ If her detectors are not clicking, the state of the field is $|0_{Alice}\rangle$, obeying $a_\omega|0_{Alice}\rangle = 0 \ \forall \omega$.

One problem though: In this cds, far away, i.e., as $r \to \infty$, the metric doesn't become the Minkowski $\eta_{\mu\nu}$.

Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cds in which:

- ☐ $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r \to \infty$.
- ☐ $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- ☐ photons travel at 45°
- ☐ we know action, K.G. equation and mode functions.

⇝ Bob's choice is the Tortoise coordinate system.

Tortoise cds (t^*, r^*):

- In terms of the Schwarzschild cds:

$$t^* := t$$

$$r^* := r - 2M + 2M \log\Big(\overbrace{\frac{r}{2M} - 1}^{\text{must require } r > 2M\,!}\Big)$$

⇒ **Important**: This is in principle invertible, to obtain

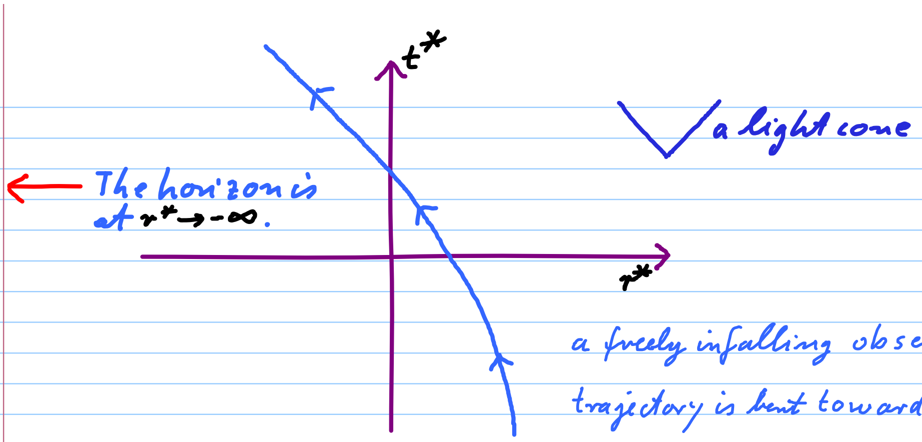
$$r = r(r^*)$$

but only for $r > 2M$!

⇒ The tortoise cds only cover the BH's outside!

Metric: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}\left(dt^{*2} - dr^{*2}\right)$$

Conformal factor →1 as $r \to \infty$, as planned but →0 at horizon.



a freely infalling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u},\bar{v})}\right)}_{\to 1 \text{ as } r \to \infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

$$\boxed{\text{Important later: } u = -4Me^{-\frac{\bar{u}}{4M}},\quad v = 4Me^{\frac{\bar{v}}{4M}}}$$

⇒ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x$$ becomes:

$$= \frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 dt^* dr^*$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\partial_{\bar{u}}\,\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\,\phi(\bar{u},\bar{v})\right) d\bar{v}\, d\bar{u}$$

⇒ Eqn of motion: $\partial_{\bar{u}}\partial_{\bar{v}}\,\phi(\bar{u},\bar{v}) = 0$

⇒ Solution for the QFT found as before:

$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^{\infty}\frac{d\omega}{(2\pi)^{1/2}}\frac{1}{(2\omega)^{1/2}}\left(e^{-i\omega\bar{u}}\,\hat{b}_\omega + e^{i\omega\bar{u}}\,\hat{b}^{+}_\omega + \text{left movers}\right)$$

obeys the 3 conditions: EoM, CCRs and hermiticity.

Bob's notion of vacuum

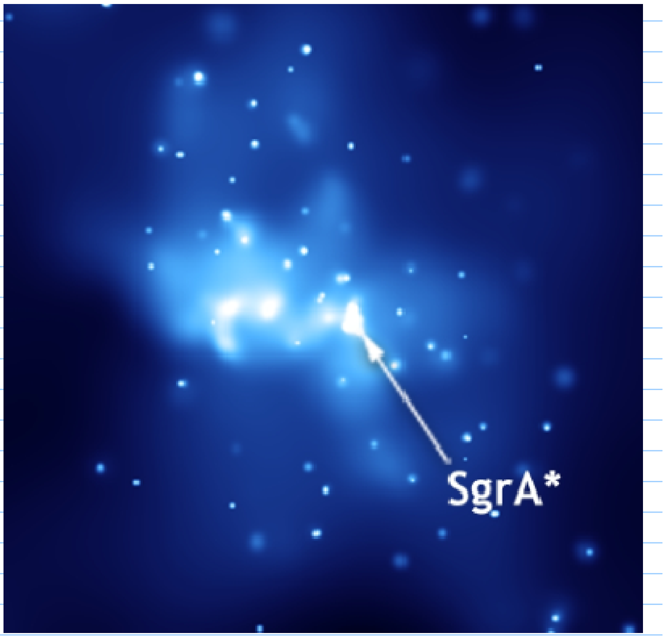
- For Bob, out at $r \to \infty$, the metric is $g_{\mu\nu} = \eta_{\mu\nu}$.
- If Bob's detectors are not clicking, the state of the field is $|0_{Bob}>$, obeying $\hat{b}_\omega|0_{Alice}> = 0\ \forall\omega$.

Modelling real black holes

E.g.: Sagittarius A^*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away at centre of Milky Way.

→ Observations coming up 2017 by Event Horizon Telescope Array (in mm band) with enough resolution to see the event horizon.



Real black holes:

- they have complicating properties, such as
  - a ring down
  - peculiar velocity
  - angular momentum
  - charges
  - and even mass changes (through absorption or emission).

Simple model:

- Let us neglect all these.
- Also assume that the rest of the universe is empty.

⇒ In good approximation the spacetime should be described by the Schwarzschild metric.

Which is then the state $|\psi\rangle$ of the quantum field?

Q: Is $|\psi\rangle = |0_{Alice}\rangle$ or perhaps $|\psi\rangle = |0_{Bob}\rangle$?

A: Probably both: $|\psi\rangle = |0_{Alice,right}\rangle \otimes |0_{Bob,left}\rangle$

Here: $a_{\omega,right}\, |0_{Alice,right}\rangle = 0 \;\forall \omega$

$b_{\omega,left}\, |0_{Bob,left}\rangle = 0 \;\forall \omega$

Why?

We cannot reliably calculate through the collapse process, because it involves tracking waves being infinitely blue-shifted at the horizon (→ see the Transplanckian problem for BHs).

Heuristic arguments yield:

- If, as we assume, the rest of the universe has no stars etc, then there should be no flux of quanta into the black hole.

  ⇒ The left-moving (i.e. ingoing) modes should be in the state

  $$|0_{Bob,left}\rangle$$

- Can the right moving (i.e. outgoing) modes be in the state $|0_{Bob,right}\rangle$?

  No!

Recall:

$$\boxed{\underset{\uparrow\text{Alice's}}{u} = -4M e^{-\frac{\bar{u}\,\leftarrow \text{Bob's}}{4M}}, \quad v = 4M e^{\frac{\bar{v}}{4M}}}$$

Compare with (from the previous lecture):

$$\boxed{\underset{\uparrow\text{inertial}}{u} = -\frac{1}{a}\, e^{-a \underset{\uparrow\text{accelerated}}{\bar{u}}}}$$

$\Rightarrow$ Alice's and Bob's cds relate in the same way as the inertial and accelerated before,

with $a = \frac{1}{4M}$

$\Rightarrow$ $|0_{Bob,right}\rangle$ has divergent vacuum energy towards the horizon!

$\Rightarrow$ If the QFT is in the state $|0_{Bob,right}\rangle$, then:

☐ Via the Einstein equation, this would contradict our assumption of spacetime being Schwarzschild (which solves

$$G_{\mu\nu}(g_{\text{Schwarzschild}}) = T_{\mu\nu} \quad \text{with } T_{\mu\nu} = 0.$$

☐ During the collapse, the quantum state will be energetically prevented to evolve into the state

$$|0_{Bob,right}\rangle$$

(in the Schrödinger picture).

☐ Alice would see a diverging amount of quantum field fluctuations and particles as she crosses the horizon.

⇒ She would be able to tell the location of the horizon by local measurements in a free-falling lab.

⇒ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

☐ Their contribution to $T_{\mu\nu}$ is smooth across the horizon.

☐ Alice does not see a burst of particles from the horizon.

A: $|0_{Alice,right}\rangle$ has these properties, (via previous lecture's results).

⇒ Plausible is that the state of the QFT is:

$$\boxed{|\psi\rangle = |0_{Alice,right}\rangle \otimes |0_{Bob,left}\rangle}$$

Q: What, therefore, should we see at rest from far?

A: Our natural cds is Bob's then.

⇒ We see no ingoing (left-moving) radiation.

But we can repeat the calculations of the previous lecture for the outgoing modes, using $a = 1/4M$

⇒ We see an outflux of quanta of temperature:

$$\boxed{T_H = \frac{1}{8\pi M}}$$

Recall: $T_H = \frac{a}{2\pi}$

Summary of Unruh-Hawking connection:

Minkowski space	Schwarzschild spacetime
Accelerated observer's vacuum: "Rindler vacuum"	Bob's vacuum: "Boulware vacuum"
Inertial observer's vacuum: "Minkowski vacuum"	Alice's vacuum: "Kruskal vacuum"

Remarks: ☐ The state $|0_{Bob,right}\rangle$ (outgoing) was disqualified due to its contribution to $T_{\mu\nu}$ which would diverge towards the horizon.

Is $|0_{Bob,right}\rangle$ having the same problem?
No, it would have that problem at the past horizon but a real black hole doesn't have one (unlike an accelerated observer.)

☐ We dropped the angles φ, θ. Do they matter?

Yes, it leads to a weakening of Hawking radiation:

The mode decomposition now involves the analog of Fourier for angles: spherical harmonics.

⇝ The Klein Gordon equation becomes:

$$\left(\Box + \underbrace{\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)}_{=:\,V_\ell(r)}\right)\phi_{\ell m}(t,r)=0$$

⇒ This effective potential needs to be overcome by Hawking radiation ⇒ Grey body factor.