The Hawking effect

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \]

Singularity: \( r = 0 \)

Horizon: \( r = 2M \)

Here, \( x = (t, r, \theta, \phi) \) are called the Schwarzschild coordinates.

Schwarzschild coordinates long confused intuition:

- The singularity at \( r = 2M \) is not real; it disappears in other coordinate systems. The curvature is smooth across \( r = 2M \).

- Due to the sign changes across \( r = 2M \), for \( r < 2M \) it is spacelike and \( dr \) is timelike for \( r < 2M \).

- Consider, for example, a traveler, Alice, who is freely falling from \( r = r_0 \) to \( r = 0 \):

\[ t = \frac{\sqrt{2}}{2} \left( \frac{1}{1 + \cos(w)} \right) \]

Here: \( 0 < t < \pi \) and \( w = \frac{\sqrt{2}}{2M} \left( d + 5w(w) \right) \)

- For quantization, and better choices of coordinate systems!
Simplification: For now, we drop the $\varphi$ and $\Theta$ coordinates.

First design of a new cds $(T, R)$ - Alice's choice (for $\tau = 2M$):

- Require $g_{\tau\tau}(T, R)$ to be regular across $\tau = 2M$.
- Require $g_{\tau\tau}(0, 0) = g_{\tau\tau}$ at $\tau = 2M$. If there's really no singularity at $\tau = 2M$ this must be possible.
- Extend this cds so that $g_{\tau\tau}(T, R) = f(T, R) g_{\tau\tau}$ because then we know:
  - the action
  - the Klein-Gordon equation
  - the solution space of the K.G. equation.
  - which is the mode form of the vacuum in this cds.

$\Rightarrow$ Alice's choice are the Kruskal-Szekeres coordinate $(T, R)$:

\begin{align*}
T(t,r) &= 4M \left| \frac{r}{2M} - 1 \right|^{\frac{1}{2}} e^{\frac{r-2M}{4M}} \left( \sinh \left( \frac{t}{4M} \right) \Theta(r-2M) + \cosh \left( \frac{t}{4M} \right) \Theta(2M-r) \right) \\
R(t,r) &= 4M \left| \frac{r}{2M} - 1 \right|^{\frac{1}{2}} e^{\frac{r-2M}{4M}} \left( \cosh \left( \frac{t}{4M} \right) \Theta(r-2M) + \sinh \left( \frac{t}{4M} \right) \Theta(2M-r) \right)
\end{align*}

This map is, in principle, invertible, to obtain $t(T, R), r(T, R)$.

The Schwarzschild metric now takes this form:

\[ ds^2 = \frac{2M}{r(T, R)} e^{-\frac{r(T, R)}{2M}} \left( dT^2 - dR^2 \right) \]

Obey all conditions!

Confined prefactor = 1 as $r = 2M$
Alice was at rest at the event horizon.

The singularity is at \( T(R) = \left( R^2 + \frac{16M^2}{e} \right)^{1/2} \) and is spacelike.

Alice's light cone coordinates:

\[
\text{Metric: } ds^2 = \frac{2M}{R(u,v)} e^{-\frac{r(u,v)}{2M}} du dv
\]

which is in Minkowski light cone

\[
\Rightarrow \text{Eqn of motion: } \partial_u \partial_v \phi(u,v) = 0
\]
Solution for the QFT found as before:

\[ \hat{\Phi}(u,v) = \int_0^\infty \frac{d\omega}{(2\pi)^{3/2}} \frac{1}{(2\omega)^{1/4}} \left( e^{-i\omega u} \hat{a}_\omega + e^{i\omega v} \hat{a}^+_\omega + \text{left movers} \right) \]

obey the 3 conditions: EoM, CCRs and hermiticity.

Alice's notion of vacuum

- For Alice, as she crosses the horizon, \( g_{\mu\nu} = g_{\mu\nu} \).
- If her detectors are not clicking, the state of the field in \( |\text{Alice}\rangle \), obeying \( \omega \hat{a} |\text{Alice}\rangle = 0 \forall \omega \).

One problem though: In this case, far away, i.e., as \( r \to \infty \), the metric doesn't become the Minkowski \( g_{\mu\nu} \).

Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cts in which:

- \( g_{\mu\nu}(x) \to g_{\mu\nu} \) as \( r \to \infty \).
- \( g_{\mu\nu}(x) = f(x) g_{\mu\nu} \) everywhere.

This is so that in his cts too

- Photons travel at 45°
- We know action, k.c. equation and mode functions.

\( \Rightarrow \) Bob's choice is the Tortoise coordinate system.
Tortoise cds ($t^*, r^*$):

- In terms of the Schwarzschild cds:

\[ t^* = t \quad \text{must require } r > 2M \]  

\[ r^* = r - 2M + 2M \log \left( \frac{r}{2M} - 1 \right) \]

\[ \Rightarrow \text{Important: This is in principle invertible, to obtain } r = r(r^*) \]

\[ \text{but only for } r > 2M ! \]

\[ \Rightarrow \text{The tortoise cds only cover the BH's outside!} \]

**Metric:**

\[ ds^2 = \left( 1 - \frac{2M}{r(r^*)} \right) \left( dt^*^2 - dr^*^2 \right) \]

(Conformal factor $\to 1$ as $r \to \infty$, as planned but $\to 0$ at horizon)

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The horizon is at $r^* \to -\infty$.

- a freely falling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates:

\[ \overline{u} := t^* - r^*, \quad \overline{v} := t^* + r^* \]

The metric is then:

\[ ds^2 = \left( 1 - \frac{2M}{\overline{u}(\overline{u}, \overline{v})} \right) d\overline{u} d\overline{v} \]

\[ \to 1 \quad \text{as } \overline{u} \to \infty \quad \text{and} \quad \to 0 \quad \text{as } \overline{r} \to 2M \]

**Important Later:**

\[ u = -4Me^{\frac{\overline{u}}{4M}}, \quad v = 4Me^{\frac{\overline{v}}{4M}} \]
The action:

\[ S[\phi] = \frac{1}{2} \int g^{\alpha \beta} \phi_{,\alpha} \phi_{,\beta} \sqrt{g} \, d^2 x \]

becomes:

\[ = \frac{1}{2} \int (\partial_{\alpha} \phi(\xi^*, \tau^*))^2 - (\partial_{\tau} \phi(\xi^*, \tau^*))^2 \, d\tau^* d\xi^* \]

\[ = \int_{-\infty}^{\infty} \int_{0}^{\infty} (\partial_{\xi} \phi(\bar{\xi}, \bar{\tau})) (\partial_{\tau} \phi(\bar{\xi}, \bar{\tau})) \, d\bar{\tau} \, d\bar{\xi} \]

\[ \Rightarrow \text{Eqn of motion:} \quad \partial_{\xi} \partial_{\tau} \phi(\bar{\xi}, \bar{\tau}) = 0 \]

Solution for the QFT found as before:

\[ \phi(\bar{\xi}, \bar{\tau}) = \int_{0}^{\infty} \frac{dw}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2\omega)^{\frac{1}{2}}} \left( e^{-i\omega \bar{\xi}} b_\omega + e^{i\omega \bar{\xi}} b_\omega^\dagger + \text{left movers} \right) \]

satisfy the 3 conditions: EOM, CCRs and hermiticity.

Bob’s notion of vacuum

- For Bob, out at \( v \to 0 \), the metric is \( g_{\mu \nu} = \eta_{\mu \nu} \).
- If Bob’s detectors are not clicking, the state of the field is \( |0_{\text{Bob}}\rangle \), obeying \( b_\omega |0_{\text{Alice}}\rangle = 0 \) \( \forall \omega \).
Modelling real black holes

E.g.: Sagittarius A*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away at centre of Milky Way.

Observations coming up 2017 by Event Horizon Telescope array (in mm band) with enough resolution to see the event horizon.

Real black holes:

- they have complicating properties, such as
  - a ring down
  - peculiar velocity
  - angular momentum
  - charges
  - and even mass changes (through absorption or emission).

Simple model:

- Let us neglect all these.
- Also assume that the rest of the universe is empty.

⇒ In good approximation the spacetime should be described by the Schwarzschild metric.
Which is then the state $|\Psi\rangle$ of the quantum field?

Q: Is $|\Psi\rangle = |\text{Alice}\rangle$ or perhaps $|\Psi\rangle = |\text{Bob}\rangle$?

A: Probably both: $|\Psi\rangle = |\text{Alice, right}\rangle \otimes |\text{Bob, left}\rangle$

Here: $|\text{Alice, right}\rangle \otimes |\text{Bob, left}\rangle = 0 \forall \omega$

Why?

We cannot reliably calculate through the collapse process, because it involves tracking waves being infinitely blue-shifted at the horizon (→ see the Transplanckian problem for BHs).

Heuristic arguments yield:

1. If, as we assume, the rest of the universe has no stars etc., then there should be no flux of quanta into the black hole.

   $\Rightarrow$ The left-moving (i.e. ingoing) modes should be in the state $|\text{Bob, left}\rangle$

2. Can the right-moving (i.e. outgoing) modes be in the state $|\text{Bob, right}\rangle$?

   No!
Recall:

\[ \frac{-\hat{x}}{4M}, \quad \frac{\hat{v}}{4M} \]

Compare with (from the previous lecture):

\[ \mu = -\frac{1}{\alpha} e^{-\alpha \mu} \]

\[ \Rightarrow \text{Alice's and Bob's cdfs relate in the same way as the inertial and accelerated before,} \]

\[ \mu = \frac{1}{4M} \]

\[ \Rightarrow |\text{Bob, right}\rangle \text{ has divergent vacuum energy towards the horizon!} \]

\[ \Rightarrow \text{If the QFT is in the state } |\text{Bob, right}\rangle, \text{ then:} \]

- Via the Einstein equation, this would contradict our assumption of spacetime being Schwarzschild (which solves 
  \[ G_{\mu\nu}(g_{\text{Schwarzschild}}) = T_{\mu\nu} \text{ with } T_{\mu\nu} = 0. \]

- During the collapse, the quantum state will be energetically prevented to evolve into the state 
  \[ |\text{Bob, right}\rangle \]
  (in the Schrödinger picture).
Alice would see a diverging amount of quantum field fluctuations and particles as she crosses the horizon.

⇒ She would be able to tell the location of the horizon by local measurements in a free-falling lab.

⇒ This would contradict the equivalence principle.

Q: What state do the right-moving (outgoing) modes have to be in, so that

- Their contribution to \( T_{\mu\nu} \) is smooth across the horizon.
- Alice does not see a burst of particles from the horizon.

A: \(|1_{\text{Alice, right}}\rangle\) has these properties, (via previous lecture's results).

⇒ Plausible is then the state of the QFT is:

\[ |147\rangle = |1_{\text{Alice, right}}\rangle \otimes |1_{\text{Bob, left}}\rangle \]

Q: What, therefore, should we see at rest from far?

A: Our natural cats is Bob’s then.

⇒ We see no ingoing (left-moving) radiation.

But we can repeat the calculations of the previous lecture for the outgoing modes, using \( a = \sqrt{4M} \)

⇒ We see an outflux of quanta of temperature:

\[ T_{\text{h}} = \frac{1}{8\pi M} \]

Recall: \( T_{\text{h}} = \frac{a}{2\pi} \)
Summary of Unruh-Hawking connection:

Minkowski space
Schwarzschild spacetime

Accerolated observer's vacuum: "Rindler vacuum"  Bob's vacuum: "Boulware vacuum"

Inertial observer's vacuum: "Minkowski vacuum"  Alice's vacuum: "Kruskal vacuum"

Remarks:

- The state \( |\text{Accelerated} \rangle \) (outgoing) was disqualified due to its contribution to \( T_{\mu \nu} \) which would diverge towards the horizon.

- Is \( |\text{Accelerated} \rangle \) having the same problem?

  - No, it would have that problem at the past horizon, but a red black hole doesn't have one (unless an accelerated observer.)

- We dropped the angles \( \phi, \Theta \). Do they matter?

  - Yes, it leads to a weakening of Hawking radiation.

  - The mode decomposition now involves the analog of Fourier for angles: spherical harmonics.

  \[ (\Box + (1 - \frac{2M}{r})\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right))\Phi_{\ell m}(t, \mathbf{r}) = 0 \]

  \[ V_{\Phi}(r) \]

  \[ \Rightarrow \text{This effective potential needs to be overcome by Hawking radiation} \Rightarrow \text{grey body factor.} \]