2nd quantization using Feynman’s path integral

- Assume a fixed spacetime is chosen and we are given its metric $g_{\mu\nu}(x)$ in some arbitrary coordinate system.

- Then, for each field $\phi(x,t)$ we can calculate its action $S[\phi,g]$, e.g., for the Klein Gordon fields:

$$S[\phi,g] = \frac{i}{2} \int_{t_1}^{t_2} \left( \frac{\partial}{\partial t} \phi \cdot \phi_{\mu\nu} - m^2 \phi_{\mu\nu} - 2 \phi^{\mu\nu} \right) \sqrt{g_{\mu\nu}} \, dx$$

- Following Feynman, we obtain a “probability amplitude”

$$\text{prob. ampl. } |\phi\rangle := \frac{i}{\sqrt{\hbar}} e^{\frac{i}{\hbar} S[\phi,g]}$$

for any possible field evolution $\phi(x,t)$ to occur.
A Note: \( w[\phi] = e^{iS[\phi]} \) is an un-normalized prob. amplitude

Therefore, \( c \) is the normalization constant.

Recall:
- Assume given un-normalized probabilities \( w(A_i) \)
  for some value \( A_i \) to be found.
- Thus, the expectation value \( \bar{A} \) is given by:

\[
\bar{A} = \frac{1}{c} \sum_i A_i w(A_i) \quad \text{where} \quad c = \sum_i w(A_i)
\]
- Thus:

\[
\bar{A} = \frac{\sum_i A_i w(A_i)}{\sum_i w(A_i)}
\]

\[\Rightarrow\] In QFT, now that we can calculate probability amplitudes, we should be able to calculate all expectation values!

**Example 1: Average field amplitude**

Note:
- The path integral is ill-defined analytically. But, algebraically it yields a consistent algorithm for calculating expectation values.
- Our results that thus are WKB and pathwise are valid which makes the path integral of \( \bar{A} \) also analytically well-defined.

The "Path Integral" i.e. integral over all variables \( \phi(x) \) for \( \phi(x) \in (-\infty, \infty) \):

\[
\bar{\Phi}(x) = \frac{\int \phi(x) e^{iS[\phi]} D[\phi]}{\int e^{iS[\phi]} D[\phi]} \quad \text{Normalisation}
\]

Assume e.g.:

\[
S[\phi] = \int \frac{1}{2} \phi(x) (\partial - m^2) \phi(x) + \frac{\lambda}{4!} \phi^4(x) \quad d^4x
\]

\[\Rightarrow \bar{\Phi}(x) = 0 \quad \text{We obtained this very quickly!} \]
Compare: How did we calculate $\hat{\phi}(x)$ previously?

$\hat{\phi}(x) = \langle 0 | \hat{\phi}(x) | 0 \rangle$

Notice that we can say more when using the path integral approach, and was quickly:

Hard to evaluate except if $\lambda = 0$. Then:

$\hat{\phi}(x) = \int e^{i\hat{H}t} \left( u^+_n(t) a_n + u^+_k(t) a^+_k \right) d^3k$

Thus:

$\hat{\phi}(x) = \langle 0 | \hat{x} a + \hat{x} a^+ | 0 \rangle = 0$

Example 2: 2-point correlation functions

$G^{(2)}(x, x') = \frac{\int \phi(x) \phi(x') e^{iS[\phi]} D[\phi]}{\int e^{iS[\phi]} D[\phi]}$

*) Meaning of $G^{(2)}(x, x')$?

It shows how much the field amplitudes at events $x$ and $x'$ are correlated over some spatial and temporal distance.

*) How to calculate $G^{(2)}(x, x')$ with old methods?

$G^{(2)}(x, x') = \langle 0 | \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle$  (A)

Here, we assume for now that $x_0 \neq x'_0$.

*) In space times where we have an explicit mode decomposition, e.g., FRW,

Recall: This means that the result depends on the identification of the vacuum state.

$\hat{\phi}(x) = \int e^{i\hat{H}t} \left( u^+_n(t) a_n + u^+_k(t) a^+_k \right) d^3k$  (B)

We obtain $G^{(2)}(x, x')$ in terms of $u^+_n(t)$.  

* Why \( x_0 \neq x_0' \)?

We have \([\hat{\phi}(x), \hat{\phi}(x')] = 0\) if \( x_0 = x_0' \)

but, in general, they don’t commute.

We choose: Earliest is always right, later is right.

* To automate the bookkeeping, define \( T \):

\[
G^{(2)}(x, x') = \langle 0 | T \hat{\phi}(x) \hat{\phi}(x') | 0 \rangle
\]

Time ordering operator

\[
= \langle 0 | \Theta(x_0 - x_0') \hat{\phi}(x) \hat{\phi}(x')
+ \Theta(x_0' - x_0) \hat{\phi}(x') \hat{\phi}(x) | 0 \rangle
\]

Heaviside step function

**Remarks:**

* From Eqs. A, B follows \( G^{(2)}(x, x') \).

* \( G^{(2)}(x, x') \) is non-zero almost everywhere!

* If \( x, x' \) are in each other’s lightcone then correlation of \( \phi(x), \phi(x') \) can be due to the propagation of the amplitude from one event, say \( x \), to the other, \( x' \).

* But \( G^{(2)}(x, x') \neq 0 \) even if \( x, x' \) are spacelike separated, e.g., if \( x_0 = x_0' \)! This cannot be due to communication, i.e., propagation. Indeed: 107 is entangled!
We saw how to calculate $G^{(2)}$ using old methods.

**How to calculate $G^{(2)}$ using the path integral?**

$$G^{(2)}(x, x') = \frac{\int \phi(x) \phi(x') e^{i S_{\phi}} D\phi}{\int e^{i S_{\phi}} D\phi}$$

**Use Dyson Schwinger method:**

Consider integral of a total derivative:

$$\int \frac{\delta}{\delta \phi(x)} \left( \phi(x') e^{i S_{\phi}} \right) D\phi$$

$= \text{boundary term}$

$= 0$

Thus:

$$0 = \int \frac{\delta}{\delta \phi(x)} \left( \phi(x') e^{i S_{\phi}} \right) D\phi$$

$$= \int \left( \delta^{\mu}(x-x') \phi(x') + \phi(x') \frac{\delta S_{\phi}}{\delta \phi(x)} \right) e^{i S_{\phi}} D\phi$$

(assume now $\lambda = 0$)  

$$= \int \delta^{\mu}(x-x') \left[ e^{i S_{\phi}} \right] \left[ \phi(x) \phi(x') \right] e^{i S_{\phi}} D\phi$$

$$\Rightarrow 0 = \delta^{\mu}(x-x') \int e^{i S_{\phi}} D\phi + i \nabla_{\alpha} \left( \nabla_{\alpha} - m^{2} \right) \phi(x') \phi(x) e^{i S_{\phi}} D\phi$$

$$\Rightarrow \int \frac{1}{\delta^{\mu}(x-x')} \frac{\delta}{\delta \phi(x)} \left( \phi(x') e^{i S_{\phi}} \right) D\phi = i \delta^{\mu}(x-x')$$

Thus, $G^{(2)}$ is called a Green's function for the Klein-Gordon equation.

$$\Rightarrow \int \frac{1}{\delta^{\mu}(x-x')} \frac{\delta}{\delta \phi(x)} \left( \phi(x') e^{i S_{\phi}} \right) D\phi = i \delta^{\mu}(x-x')$$
Exercise: Show that indeed \(T \hat{\phi}(x, \lambda \cdot m^2) \langle 0 | T \hat{\phi}(x') \hat{\phi}(x') | 0 \rangle = \delta'(x-x')\)

Hint: \((x, \lambda \cdot m^2) \hat{\phi}(x)\) so of course, but \(x\) in \(T\) adds momentarily on \(O(t', t') \in T\).

Note:
- For any given metric \(g\) this becomes an explicit partial differential equation.
- If we can solve it, we know all about the propagation and also all about the correlations caused by the entanglement of the vacuum state.
- But is the solution unique?

Does the operator \((\Delta_{x} - m^2)\) have a unique eigenvector?

No! Because 3 solutions, to \((\Delta_{x} - m^2) G^{(a)}_{km}(x, x') = 0\):

Example Minkowski space:

Fourier transform spatial coordinates, to obtain:

\[
(\partial_{x}^2 + \partial_{t}^2 + m^2) \ G^{(3)}(t, t', k) = i \delta(t-t')
\]

Thus, \(G^{(3)}(t, t', k)\) is unique only up to the choice of a solution to

\[
(\partial_{x}^2 + \partial_{t}^2 + m^2) \ G^{(a)}_{km}(t, t', k) = 0
\]

These are the mode solutions:

\[
G^{(2)}_{km}(t, t', k) = \phi_{k}(t-t')
\]

\(\Rightarrow\) The ambiguity in \(G^{(a)}\) corresponds to the ambiguity in fixing the vacuum state.
The central quantities in elementary particle physics:

- Of central significance in quantum field theory are the fields’ n-point correlation functions:

\[ G(x_1, x_2, ..., x_n) = \int \phi(x_1) \phi(x_2) ... \phi(x_n) e^{iS[\phi]} \]  

\[ \text{all } \phi \]  

with, as before: \( \nu^i = \left\{ e^{iS[\phi]} \right\} \) 

- Here:
  * Each \( x_i \) is a point in spacetime, i.e., an event.
  * The integral is formally the sum over all (differentiable, square integrable etc.) fields \( \phi \).

Proposition:  \[ G^{(n)}(x_1, ..., x_n) = \langle 0| T\hat{\phi}(x_1) ... \hat{\phi}(x_n) | 0 \rangle \]  

Proof strategy: Show that LHS and RHS obey same differential equation.

- Case \( n = 2 \):

\[ \nabla_j \left( \partial_x + m^2 \right) G^{(2)}(x, y) = i \delta^4(x-y) \]  

(we show)

\[ \nabla_j \left( \partial_x + m^2 \right) \langle 0| T\hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = i \delta^4(x-y) \]  

(given as exercise)

- For general \( n \): "Schrödinger–Dyson equations"

From path integral, easy to derive, generalizing above ansatz which worked for \( n = 2 \):

\[ 0 = \int \frac{\delta}{\delta \phi(x)} \left( \phi(x_1) ... \phi(x_n) e^{iS[\phi]} \right) D[\phi] \]  

... then work out the derivatives.
It is much more work in the operator formalism to derive that the \( \langle 0 | \mathcal{O}(T) \cdots \mathcal{O}(x) | 0 \rangle \) obey the same equations. But it can be done.

Why cumbersome in operator formalism?

Recall how \( \mathcal{O}_1^{\infty} \) in \( T \) affects \( \mathcal{O}_1^{\infty} \) in \( \langle 0 | \mathcal{O}(T) \cdots \mathcal{O}(x) | 0 \rangle \).

Note:
In the "imaginary time" formulation, the KG eqn is elliptic, thus has no homogeneous solutions \( \mathcal{O}^{\infty} \) unique in PI approach. Thus, analytical continuation to real time yields a unique choice — and generally the right choice. Problem: changing time function not generally evaluable on curved spacetime.

Thus, \( \mathcal{O} + T \) in operator and PI formalism yield same predictions for the correlation functions (and everything else), provided the Dyson Schwinger equations are solved with the same initial conditions.

(i.e., same vacuum state)

Notice: PI approach does not fix the initial conditions (and thus the vacuum state).

What PI approach is best for: Perturbation theory for interactions

\* We introduce an auxiliary field, \( J \), called a "source field":

\[
G(x_1, x_2, \ldots, x_n) := N \int e^{\frac{i}{\hbar} \mathcal{S}[\phi]} \Delta \phi \]

\[
= ( -i )^{n-1} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \int e^{\frac{i}{\hbar} \mathcal{S}[\phi] + \frac{i}{\hbar} \int_j \mathcal{S}[\phi]_j \frac{d x_j}{\hbar} } \]

\[
= \mathcal{S}[\phi, J]
\]

\* Let us define an action, \( \mathcal{S}[\phi, J] \), that includes the sources:

\[
\mathcal{S}[\phi, J] := \int \frac{1}{2} \phi(x) \left( \frac{\partial^2}{\partial x^2} + m^2 \right) \phi(x) + \frac{1}{2} \phi(x)^2 + J(x) \phi(x) \ dx
\]
**Definition:**

The "partition function", $\mathcal{Z}[J]$, is defined as:

$$
\mathcal{Z}[J] = N \int_{\mathcal{M}} e^{i \tilde{S}[\phi, J]} \mathcal{D}[\phi]
$$

**We can now write the correlation functions in this form:**

$$
G(x_1, \ldots, x_n) = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \mathcal{Z}[J] \bigg|_{J=0}
$$

**How can we now calculate the $n$-point functions?**

*To this end, it obviously suffices to calculate $\mathcal{Z}[J]$, since $\mathcal{Z}[J]$ is "generating functional" for the $G(x_1, \ldots, x_n)."

*Explicitly, e.g., for $\lambda \phi^4$-theory:

$$
\mathcal{Z}[J] = N \int_{\mathcal{M}} e^{i \tilde{S}[\phi, J]} \mathcal{D}[\phi]
$$

$$
= N \int e^{i \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{\hbar} \frac{\delta^2}{\delta J(x_1)} \cdots \frac{\delta^2}{\delta J(x_n)} \right)^n \phi(x_1) \cdots \phi(x_n) + \frac{\lambda}{2} \phi(x)^4 + J(x) \phi(x) \mathcal{D}[\phi]}
$$

Note: this step is also analytically ill defined. $ightarrow$

$$
= e^{i \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{\hbar} \frac{\delta^2}{\delta J(x_1)} \cdots \frac{\delta^2}{\delta J(x_n)} \right)^n \phi(x_1) \cdots \phi(x_n)} N \int e^{i \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{\hbar} \frac{\delta^2}{\delta J(x_1)} \cdots \frac{\delta^2}{\delta J(x_n)} \right)^n \phi(x_1) \cdots \phi(x_n) + J(x) \phi(x) \mathcal{D}[\phi]}
$$
* Introduce a shorthand matrix & vector notation:

\[
\int_{\mathbb{R}^4} \frac{1}{2} \phi(x) \left( \frac{\delta^4}{\delta \phi^2} - \Delta + m^2 \right) \phi(x) d^4 x + \int_{\mathbb{R}^4} J(x) \phi(x) d^4 x
\]

With suitable choice of a countable basis in the function space of the fields:

\[
= \sum_{i,j} \frac{1}{2} \phi_i M_{ij} \phi_j + \sum_k \phi_i \bar{f}_k
\]

* Thus:

\[
\mathcal{Z}[J] = e^{i \int \frac{1}{2} \left( \frac{\delta}{\delta \phi} \right)^4 d^4 x N \int e^{i \left( \frac{1}{2} \phi(x) \left( \frac{\delta^2}{\delta \phi^2} - \Delta + m^2 \right) \phi(x) + J(x) \phi(x) d^4 x
\]

\[
\mathcal{Z}[J] = e^{i \frac{1}{8} \mathbb{S}_3^4 N \int e^{i \left[ \frac{1}{2} \phi(x) \left( \frac{\delta^2}{\delta \phi^2} - \Delta + m^2 \right) \phi(x) + J(x) \phi(x) d^4 x
\]

* We observe: (completion of squares)

\[
\frac{1}{2} \phi M \phi + J \phi = \frac{1}{2} (\phi + \Delta \phi) M (\phi + \Delta \phi) - \frac{1}{2} J \phi M \phi
\]

* Thus:

\[
\mathcal{Z}[J] = e^{i \frac{1}{8} \mathbb{S}_3^4 N \int e^{i \left[ \frac{1}{2} (\phi + \Delta \phi) M (\phi + \Delta \phi) - \frac{1}{2} J \phi M \phi \right] d^4 x
\]

change integration variable: \( \tilde{\phi} = \phi + \Delta \phi \Rightarrow \)

\[
= e^{i \frac{1}{8} \mathbb{S}_3^4} e^{i \frac{1}{2} J \tilde{\phi} M \tilde{\phi}} N \int e^{i \frac{1}{2} \tilde{\phi} M \tilde{\phi}} d^4 x
\]

\[
= \mathcal{N}^e i \frac{1}{8} \mathbb{S}_3^4 e^{i \frac{1}{2} J \tilde{\phi} M \tilde{\phi}}
\]

\( \mathcal{N} \) Normalization constant
Back in original notation:

\[ Z[J] = \tilde{N} \int \frac{d^4x}{(2\pi)^4} e^{\frac{i}{\hbar} \int_{\cal R} J(x) F(x) dx} \]

Recall that:

\[ M = \frac{\lambda}{2} - \Delta + m^2 \]

Thus:

\[ K = M^{-1} \text{ is a Green's function.} \]

J.e., \( K \) obeys:

\[ \left( \frac{\lambda}{2} - \Delta + m^2 \right) K(x, x') = \delta^4(x - x') \]

Here, \( K(x, x') \) is called the propagator.

\( K(x, x') \) will be represented by an edge \( x \rightarrow x' \).

Where graphs come in:

The generating function

\[ Z[J] = \tilde{N} \int \frac{d^4x}{(2\pi)^4} \left( \frac{\delta}{\delta J(x)} \right)^4 e^{\frac{i}{\hbar} \int_{\cal R} J(x) F(x) dx} \]

\[ = \tilde{N} \left( 1 + \frac{i}{\hbar} \int \frac{\delta}{\delta J(x)} \left( \int_{\cal R} J(x') F(x) dx' \right) dx + ... \right) \left( 1 - \frac{i}{\hbar} \int \frac{\delta}{\delta J(x)} \left( \int_{\cal R} J(x') F(x) dx' \right) dx' \right) + ... \]

may now be viewed as a sum of graphs (all x's, etc integrated)

\[ \int d^4x K(x, x')^2 \]

\[ = \tilde{N} \left[ 1 - \frac{1}{2} \left( \frac{\lambda}{2} \right)^2 + \frac{1}{2!} \left( \frac{\lambda}{2} \right)^4 \left( \frac{c_2}{x^2} \right)^4 + \frac{1}{3!} \left( \frac{\lambda}{2} \right)^6 \left( \frac{c_2}{x^2} \right)^6 + ... \right. \]

\[ + \frac{i}{\hbar} \left( \frac{\lambda}{2} \right)^2 \left( \frac{c_2}{x^2} \right)^2 \left( \frac{\lambda}{2} \right)^4 \left( \frac{c_2}{x^2} \right)^4 + \frac{i}{\hbar} \left( \frac{\lambda}{2} \right)^6 \left( \frac{c_2}{x^2} \right)^6 + ... \left. + ... \right) + ... \]
* We have:

- One kind of edge:
  \[ \overrightarrow{e} = \kappa(x, y) \quad \text{"propagator"} \]

- Two kinds of vertices:
  \[ x = \frac{i}{2} \gamma(x) \quad \text{A free end: an ingoing or outgoing particle} \]
  \[ x \quad \frac{i}{\hbar} \delta(\gamma(x) - x_1) \delta(\gamma(x) - x_2) \delta(\gamma(x) - x_3) \delta(\gamma(x) - x_4) \]
  This vertex describes the collision of particles

* Note: \( Z[J] \) contains connected and disconnected graphs.

* Definition: Let \( i W[J] \) be the sum of only all connected graphs.

* Proposition: We have:
  \[ Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \left( i W[J] \right)^N = e^{i W[J]} \]
  This is clear because:
  - disconnected graphs are products of connected graphs
  - the factor \( 1/N! \) avoids overcounting because in \( Z[J] \) the order of the connected subgraphs does not matter (since the vertices can be \( \mathbb{R} \)-labeled).

* Thus: \( W[J] = -i \ln Z[J] \) is the generating functional for the connected graphs.
**Outlook:**

Why work with $W[J]$, instead of $Z[J] = e^{-W[J]}$?

Recall: $Z[J] = \sum_{\hat{\mathcal{G}}} e^{-\hat{S}[\hat{\mathcal{G}}, J]} D[J]$ i.e.: $Z[0] = 1$ means $N^* = \sum_{\text{all graphs without end vertices}}$, i.e., it is the sum over all disconnected graphs.

Thus:

\[
(i)^{n} \frac{\delta}{\delta J(x_{1})} \ldots \frac{\delta}{\delta J(x_{n})} \frac{Z[J]}{J = 0}
\]

inhibits $N$ and the disconnected graphs but in

Exercise: Show that $(\ln f)^{i} = \frac{i}{f}$

\[
(i)^{n} \frac{\delta}{\delta J(x_{1})} \ldots \frac{\delta}{\delta J(x_{n})} \frac{W[J]}{J = 0}
\]

they cancel!