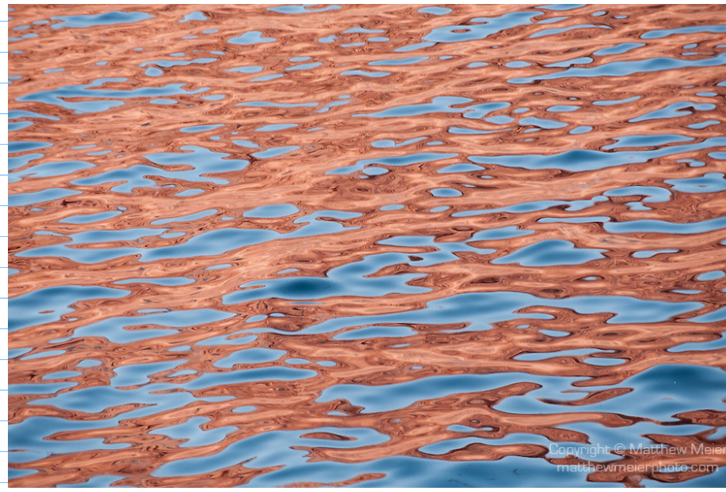
Note Title

# QFT for Cosmology, Achim Kempf, Winter 2016, Lecture 3




- Quantization conditions:

analogous to:

$$[\hat\phi(x,t),\hat\pi(x',t)] = i\hbar\,\delta^3(x-x') \qquad\qquad [\hat q_a(t),\hat p_{a'}(t)] = i\hbar\,\delta_{aa'}$$

$$[\hat\phi(x,t),\hat\phi(x',t)] = 0 \qquad\qquad [\hat q_a(t),\hat q_{a'}(t)] = 0$$

$$[\hat\pi(x,t),\hat\pi(x',t)] = 0 \qquad\qquad [\hat p_a(t),\hat p_{a'}(t)] = 0$$

- We keep the equations of motion:

$$\dot{\hat\phi}(x,t) = \hat\pi(x,t) \qquad \text{(E1)} \qquad \dot{\hat q}_a(t) = \hat p_a(t)$$

$$\dot{\hat\pi}(x,t) = -(-\Delta+m^2)\,\hat\phi(x,t) \qquad \text{(E2)} \qquad \dot{\hat p}_a(t) = -K_a\hat q_a(t)$$

- Note: $\phi^*(x,t)=\phi(x,t)$ now implies hermiticity: $\hat\phi^\dagger(x,t)=\hat\phi(x,t)$

□ Proposition:

$E1, E_2$ follow from the Heisenberg eqns

analogous to:

$$i\hbar\, \dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$$

$$i\hbar\, \dot{\hat{q}}_a(t) = [\hat{q}_a(t), \hat{H}]$$

$$i\hbar\, \dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}]$$

$$i\hbar\, \dot{\hat{p}}_a(t) = [\hat{p}_a(t), \hat{H}]$$

with this QFT Hamiltonian:

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)\left(m^2 - \Delta\right)\hat{\phi}(x',t)\, d^3x'$$

$$\hat{H} = \sum_a{}' \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2}{2}\hat{q}_a^2$$

Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields ✓
3. 2nd quantization ✓
4. Harmonic oscillators in fields ⇒ vacuum fluctuations

## 4. Harmonic oscillators in quantum fields

□ From the above, we need to solve 2 equations:

a.) The K.G. eqn: $\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\hat{\phi}(x,t) = 0$

$x = (x_1, x_2, x_3)$

b.) The commutation rels: $[\hat{\phi}(x,t), \dot{\hat{\phi}}(x',t)] = i\hbar\,\delta(x-x')$

- **Q:** How to solve these eqns?

  **A:** Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

**1st Difficulty:** (in reducing the QFT problem to harmonic oscillators)

- In the K.G. equation,

$$\dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \quad \xleftrightarrow{\text{Analogy}} \quad \dot{\hat{p}}_a(t) = -K_a\,\hat{q}_a(t)$$

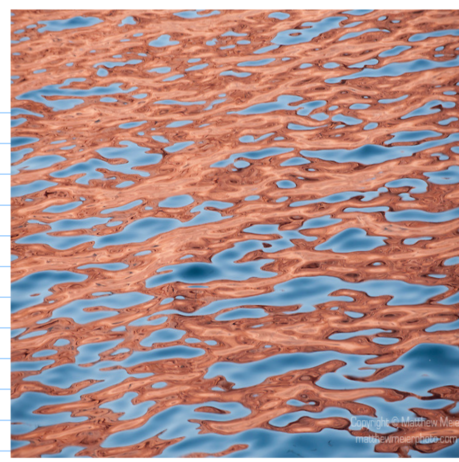
we notice that $(-\Delta + m^2)$, unlike $K_a$, is not a number!

- **Q:** Can we "transform" $(-\Delta + m^2)$ into a number?

  **A:** Yes: Fourier transform turns derivatives into numbers!

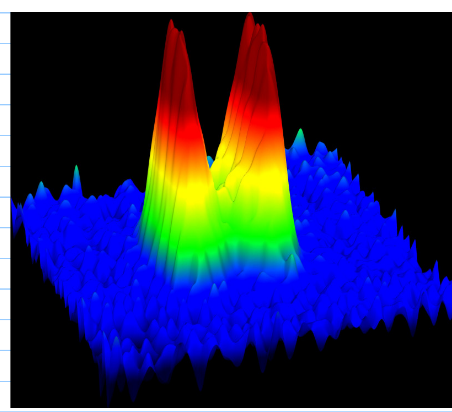
The local field oscillators are coupled.

$\Rightarrow$ Excitations spread.



The oscillators that are local in momentum space are uncoupled.

$\Rightarrow$ Excitations don't spread in momentum space.



## Fourier transform of the spatial variables $x_i$:

### Definition:

$xk = \sum_{i=1}^{3} x_i k_i$ ; $k = (k_1, k_2, k_3)$

$$\hat\phi(k,t) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ixk}\, \hat\phi(x,t)\, d^3x$$

Traditional notation: $\hat\phi_k(t) := \hat\phi(k,t)$

Traditional terminology: $\hat\phi_k(t)$ is called the field's **k-mode**.

### Inverse Fourier transform:

$$\hat\phi(x,t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ixk}\, \hat\phi_k(t)\, d^3k$$

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### Proposition: (Exercise: show this)

Analogous to:

a) $k^2 = \sum_{i=1}^{3} k_i^2$

$$\hat H = \int_{\mathbb{R}^3} \frac{1}{2}\hat\pi_k^+(t)\,\hat\pi_k(t) + \frac{1}{2}\,\hat\phi_k^+(t)\,(k^2+m^2)\,\hat\phi_k(t)\; d^3k$$

$\hat H = \sum_a \frac{1}{2}\hat p_a \hat p_a + \frac{1}{2}\omega_a \hat q_a \hat q_a$

b) $[\hat\phi_k(t), \hat\pi_{k'}(t)] = i\hbar\, \delta^3(k+k')$ $\qquad$ $[\hat q_a, \hat p_{a'}] = i\hbar\,\delta_{a,a'}$

$[\hat\phi_k(t), \hat\phi_{k'}(t)] = 0$ $\qquad$ $[\hat q_a(t), \hat q_{a'}(t)] = 0$

$[\hat\pi_k(t), \hat\pi_k(t)] = 0$ $\qquad$ $[\hat p_a(t), \hat p_{a'}(t)] = 0$

c) $\dot{\hat\phi}_k(t) = \hat\pi_k(t)$ $\qquad$ $\dot{\hat q}_a(t) = \hat p_a(t)$

$\dot{\hat\pi}_k(t) = -(k^2+m^2)\,\hat\phi_k(t)$ $\qquad$ $\dot{\hat p}_a(t) = -\omega_a^2\, \hat\phi_a(t)$

$\Rightarrow$ For each mode $\vec k$ we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2+m^2}$.

- **Exercise:**

Show that a) + b) + Heisenberg eqn $\dot{\hat{f}}(t) = \frac{1}{i\hbar}[\hat{f}(t), \hat{H}]$ yields c.)

($\hat{f}$ is arbitrary. E.g. $\hat{f} = \hat{\phi}_k$ or $\hat{f} = \hat{\pi}_k$)

## $2^{nd}$ Difficulty: (in reducing the QFT problem to harmonic oscillators)

- We notice that the commutation relations

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta^3(k+k') \quad \text{and} \quad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\, \delta_{a,a'}$$

do not match, because the Kronecker $\delta$ is only either 0 or 1, unlike the Dirac $\delta$!

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- **Idea:** If use Fourier series instead, should have discrete values of $k$, thus Kronecker $\delta$ for CCR!

- **Strategy:**

1. Put system into a large box $[-L/2, L/2]^{\times 3}$
2. Assume (for example) periodic boundary conditions.

   (If box large enough it should not matter here what happens at the boundary of the box)
3. Instead of Fourier transform, we can now use Fourier series.

**Terminology:** Putting a system in a box is called "Infrared regularization".

↑ because "long" wavelengths are removed.

## Infrared regularization:

* $(k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$ with $n_1, n_2, n_3 \in \mathbb{Z}$
* $V = L^3$ (Volume of box)
* Fourier series expansion coefficients:

$$\hat{\phi}_k(t) = V^{-1/2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \hat{\phi}(x,t)\, e^{-ixk}\, d^3x$$

* The inverse is the Fourier series:

$$\hat{\phi}(x,t) = V^{-1/2} \sum_{k} \hat{\phi}_k(t)\, e^{ixk}$$

discrete set of vectors!

## The QFT problem in the box:

analogous to

a) $\hat{H} = \sum_k \frac{1}{2} \hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2} \omega_k^2 \hat{\phi}_k^+ \hat{\phi}_k$ $\qquad \hat{H} = \sum_a \frac{1}{2} \hat{p}_a \hat{p}_a + \frac{1}{2} \omega_a^2 \hat{q}_a \hat{q}_a$

$\omega_k^2 = k^2 + m^2$

Kronecker $\delta$

b) $[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\, \delta_{k,-k'}$ $\qquad [\hat{q}_a, \hat{p}_{a'}] = i\hbar\, \delta_{a,a'}$

$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$ $\qquad [\hat{q}_a(t), \hat{q}_{a'}(t)] = 0$

$[\hat{\pi}_k(t), \hat{\pi}_k(t)] = 0$ $\qquad [\hat{p}_a(t), \hat{p}_{a'}(t)] = 0$

c) $\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$ $\qquad \dot{\hat{q}}_a(t) = \hat{p}_a(t)$

$\dot{\hat{\pi}}_k(t) = -(k^2 + m^2)\, \hat{\phi}_k(t)$ $\qquad \dot{\hat{p}}_a(t) = -\omega_a^2\, \hat{\phi}_a(t)$

## $3^{rd}$ Difficulty: (in reducing the QFT problem to harmonic oscillators)

### Hermiticity:

We notice that $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$, $\hat{\pi}^+(x,t) = \hat{\pi}(x,t)$ implies

$$\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t), \quad \hat{\pi}_k^+(t) = \hat{\pi}_{-k}(t) \qquad \text{(H)}$$

(Indeed:
$\hat{\phi}_k^+(t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\hat{\phi}^+(x,t)d^3x = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ixk}\hat{\phi}(x,t)d^3x = \hat{\phi}_{-k}(t)$)

But eqns (H) do not match:

$$\hat{q}_a^+(t) = \hat{q}_a(t) \qquad \hat{p}_a^+(t) = \hat{p}_a(t)$$

**Namely:** Our $\hat{\phi}_k, \hat{\pi}_k$ are not hermitean!

### Correspondingly:

The analogy between

$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\,\delta_{k,-k'}$ and $[\hat{q}_a, \hat{p}_{a'}] = i\hbar\,\delta_{a,a'}$

suffers from $\delta_{k,-k'}$ instead of $\delta_{k,k'}$. (we do have $[\hat{\phi}_k(t), \hat{\pi}_{k'}^+(t)] = i\hbar\,\delta_{k,k'}$)

### Mukhanov:

Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators but with complex, i.e., non hermitean amplitudes.

## Proper treatment:

- Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

  Eqns of motion: $\hat{p}_k = \dot{\hat{q}}_k, \quad \dot{\hat{p}}_k = -\omega_k^2 q_k$

  Canon. com. rels: $[\hat{q}_k, \hat{p}_{k'}] = i\,\delta_{k,k'}$

  Hermiticity: $\hat{q}_k^+ = \hat{q}_k \quad \hat{p}_k^+ = \hat{p}_k$

- Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2}\left(\hat{q}_k + \hat{q}_{-k}\right) + \frac{i}{2\omega_k}\left(\hat{p}_k - \hat{p}_{-k}\right) \qquad \text{(A)}$$

**Remark:** In practice, it'll be more convenient to work with $a_k, a_k^+$:

With $a_k := \sqrt{\omega_k}\,\hat{x}_k + \frac{i}{\sqrt{\omega_k}}\hat{p}_k$ the ansatz reads: $\hat{\phi}_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a^+_{-k})$

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Exercise!

- Now, show that ansatz (A) succeeds, i.e., that indeed:

  Hamiltonian $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{1}{2}\omega_k^2 \hat{q}_k^2 \qquad \text{(H)}$

  Eqns of motion: $\hat{\pi}_k = \dot{\hat{\phi}}_k, \quad \dot{\hat{\pi}}_k = -\omega_k^2 \hat{\phi}_k$

  Canon. com. rels: $[\hat{\phi}_k, \hat{\pi}_{k'}] = i\,\delta_{k,-k'}$

  Hermiticity cond.: $\hat{\phi}_k^+ = \hat{\phi}_{-k}, \quad \hat{\pi}_k^+ = \hat{\pi}_{-k}$

- Finally, via inverse Fourier series, show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{V}}\sum_k \left\{\cos(xk)\,\hat{q}_k - \frac{1}{\omega_k}\sin(xk)\,\hat{p}_k\right\} \qquad \text{(B)}$$

**Remark:** Ansatz (A) was not unique!

The x's and p's could be more mixed! (H) could be different!

## Significance of non-uniqueness?

"No particles state"

* Ground state of $q_k, p_k$ oscillators $\rightsquigarrow$ Vacuum
* This need not be lowest energy state of the QFT Hamiltonian

→ Problem of vacuum identification on curved space.

→ See later.

**For now:** We solved, using (A), the QFT eqns of the K.G. field.

Namely, we have now solved:

$$\text{Eqns of motion:} \begin{cases} \dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \\ \dot{\hat{\pi}}(x,t) = -(-\Delta + m^2)\,\hat{\phi}(x,t) \end{cases}$$

Hermiticity: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)\,, \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t)$

Can. com. rels: $[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\,\delta^3(x-x')$

## Example: How to calculate quant. fluct. of K.G. field?

1. Solve the system of $\infty$ many quantum harmonic oscillator degrees of freedom

$$\hat{q}_k\,,\ \hat{p}_k$$

with $\hat{H} = \sum_k \frac{1}{2}\hat{p}_k^2 + \frac{\omega_k^2}{2}\hat{q}_k^2\,, \qquad \omega_k = \sqrt{k^2+m^2}$

for all $k = (k_1,k_2,k_3) = \frac{2\pi}{L}(n_1,n_2,n_3)$ where $n_1, n_2, n_3 \in \mathbb{Z}$

2. Choose a state $|\psi\rangle$ of that quantum system.

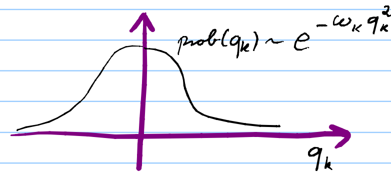
**Example:** ▫ The oscillators could all be in their lowest energy state.

We preliminarily call this $|\psi\rangle$ the vacuum state $|\psi_0\rangle$.

3. Given a state $|\psi\rangle$, we can calculate the probability (amplitude density) for finding arbitrary values $q_k(t)$, $p_k(t)$.

Example:

In vacuum state, we know that the probability distribution of the $\hat{q}_k$ (and $\hat{p}_k$ as well) is gaussian:




4. Given $|\psi\rangle$, calculate the probability distribution of the Fourier coefficients:

$$\hat{\phi}_k, \hat{\pi}_k$$

Can do because they are simply linear combinations of the harmonic oscillator variables $\hat{q}_k, \hat{p}_k$. (Exercise: calculate)

Example:

For $|\psi_0\rangle$, since $q_k, p_k$ are gaussian distributed, also the $\phi_k, \pi_k$ are gaussian distributed:

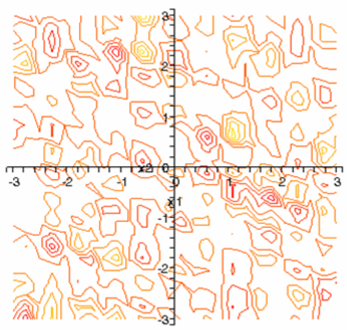
$prob(\phi_k) \sim e^{-\omega_k \phi_k^* \phi_k}$ (straightforward but tedious to show)

5. Given the prob. distribution of the $\phi_k$, use Fourier to obtain prob. distribution of $\phi(x)$!

Example:

- Consider $|\Psi_0\rangle$.
- Draw a field $\phi(x)$ from the above calculated probability distribution for fields $\phi(x)$.

Contour lines of a typical $\phi(x,t)$ drawn from the vacuum's probability distribution for $\phi$'s.



← Actual draw from that distribution.

The fluctuations trace back to the Fourier coefficients and to the $\hat{q}_k, \hat{p}_k$ which fluctuate even in lowest energy state.