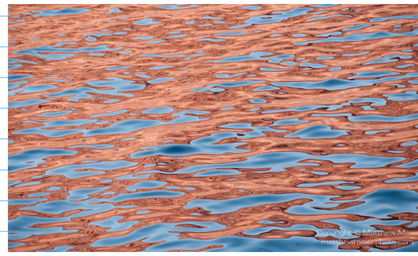
From Heisenberg to Schrödinger picture

Water:

$\phi(x,t)$



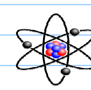
Probe amplitudes, e.g., with a cork:

Quantum field:

$\hat{\phi}(x,t)$

How to visualize an operator-valued field ?

Probe amplitudes, e.g., with atoms (lecture 8):



For now...

Assume we have some means to measure

$$\hat{\phi}(x,t)$$

at a time t for all $x \in \mathbb{R}^3$.

Q: Why possible in principle?

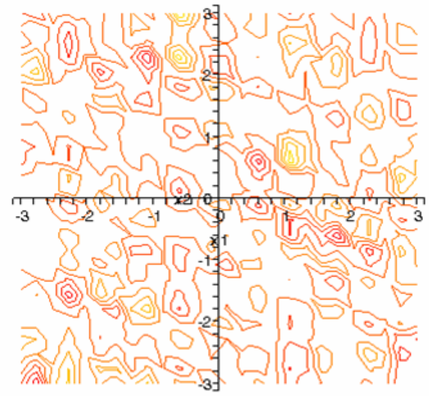
A: Because $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \quad \forall x, x' \in \mathbb{R}^3$

Note: The $\hat{\phi}(x,t)\ \forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.

$\Rightarrow$ At each x we obtain a real-valued measurement outcome, say $f(x)$.

In vacuum, a typical measurement outcome $f(x)$ is:

Shown are the level curves.



The measurement collapsed the system into the new state

$$|f\rangle \in \mathcal{H}$$

which is joint eigenstate of all $\hat{\phi}(x,t)$:

$$\hat{\phi}(x,t)\,|f\rangle = f(x)\,|f\rangle$$

Here: If $f: \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function, we denote by

$$|f\rangle \in \mathcal{H}$$

the joint eigenvector of all $\hat{\phi}(x,t)$ with eigenvalues $f(x)$:

unique up to a phase

i.e. for all $x \in \mathbb{R}^3$

$$\hat{\phi}(x,t)|f\rangle = f(x)\,|f\rangle \qquad \text{for all } x \in \mathbb{R}^3$$

Hilbert basis: The set

$$\{|f\rangle\}$$

of all joint eigenvectors of the $\hat{\phi}(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis" of $\mathcal{H}$. (up to functional analytic subtleties).

$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:

$$|\Psi\rangle = \int_{L^2(\mathbb{R}^3)} |f\rangle\langle f|\Psi\rangle$$

it's more subtle really

analogous to:

$$|\psi\rangle = \int |\vec{x}\rangle \underbrace{\langle \vec{x}|\psi\rangle}_{\psi(\vec{x})} d^3x$$

The "Wave functional"

Recall QM:

- Assume $\{\hat{R}_i\}_{i=1}^{N}$ is compl. set of commuting observables, with joint eigenvectors $|r\rangle$ obeying: $\hat{R}_i\,|r\rangle = r_i\,|r\rangle$.
- Then the function ψ, given by $\psi(r) = \langle r|\psi\rangle$ is called the "**wave function**" of $|\psi\rangle$ in the $\{\hat{R}_i\}$ basis.

Example: $\{\hat{p}_i\}$ yield mom. wave functions $\psi(p) = \langle p|\psi\rangle$, $p = \{p_1, p_2, \dots, p_N\}$

In QFT: E.g., $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ is compl. set of com. observables (or, e.g., also the $\{\hat{\pi}(x)\}$.)

with joint eigenvectors $|f\rangle$ obeying $\hat{\phi}(x)\,|f\rangle = f(x)\,|f\rangle$.

- Then, Ψ, given by

$\{|f\rangle\}$ form field ON eigen basis

(Convention: square bracket because argument is a function)

$$\Psi[f] := \langle f|\Psi\rangle$$ is called the "**wave functional**".

(Ψ called a "Functional" because argument is a function)

alternatively could use e.g. joint eigenbasis of the $\hat{\pi}(x,t)$.

Interpretation of $\Psi[f]$?

- Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{H}$ at t. (e.g., vacuum $|\Psi_0\rangle$)
- If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $f(x)$?

Answer: $\text{prob}[|\Psi\rangle \to |f\rangle] = |\langle f|\Psi\rangle|^2 = |\Psi[f]|^2$

Q: The eqn. of motion for $\Psi[f,t]$?

A: The QFT Schrödinger equation!

- For every quantum theory, we have in the Schrödinger picture of the time evolution:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

- Which form does it take for $\Psi[f,t]$?

- Here in QFT:

now independent of time!

$$\hat{H} = \int \frac{1}{2}\left(\hat{\pi}^2(x) + \hat{\phi}(x)(-\Delta + m^2)\hat{\phi}(x)\right) d^3x$$

- But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[f,t]$?

(Exercise: check)

- A valid representation of $[\hat{\phi}(x), \hat{\pi}(x')] = i\delta^3(x-x')$ is:

$$\hat{\phi}(x).\ \Psi[f,t] = f(x)\,\Psi[f,t]$$

$$\hat{\pi}(x).\ \Psi[f,t] = -i\frac{\delta}{\delta f(x)}\,\Psi[f,t]$$

functional derivative, as in variational principle used to derive Euler Lagrange equations.

- Therefore:

$$\hat{H} = \int \frac{1}{2}\left(-\frac{\delta^2}{\delta f^2(x)} + f(x)(-\Delta + m^2)f(x)\right)$$

inconvenient

- It is more convenient to use infrared-regularized momentum space:

- We now need to represent

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k,-k'}$$

on the wave functionals $\Psi[\tilde{f}, t]$. ($\tilde{f}_k$ is Fourier transform of $f(x)$)

- As you should verify, this works:

$$\hat{\phi}_k \cdot \Psi[\tilde{f}, t] = \tilde{f}_k \Psi[\tilde{f}, t]$$

$$\hat{\pi}_k \cdot \Psi[\tilde{f}, t] = -i \frac{\partial}{\partial \tilde{f}_{-k}} \Psi[\tilde{f}, t]$$

Note: Ordinary derivatives here because set of variables $\{\tilde{f}_k\}$ is discrete, since $k = \frac{2\pi}{L}(n_1, n_2, n_3)$, $\vec{n} \in \mathbb{Z}^3$.

Schrödinger equation:

$i\partial_t |\psi\rangle = \hat{H} |\psi\rangle$ becomes:

$$i\partial_t \Psi[\tilde{f}, t] = \sum_k \frac{1}{2}\left(-\frac{\partial}{\partial \tilde{f}_k}\frac{\partial}{\partial \tilde{f}_{-k}} + (k^2 + m^2)\tilde{f}_k \tilde{f}_{-k}\right) \Psi[\tilde{f}, t]$$

Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\psi(x,t) = N e^{-\frac{1}{2}\omega x^2 - i\omega t}$$

Exercise: check it. Can you solve for excited states?

Ground state solution in QFT reads, similarly:

$$\Psi[\tilde{f}, t] = N e^{-\sum_k \left(\frac{1}{2}\omega_k \tilde{f}_k \tilde{f}_{-k} - i\omega_k t\right)}$$

$\omega_k = (\vec{k}^2 + m^2)^{1/2}$

Exercise: verify

... which we had already claimed before.

Generic wave functionals

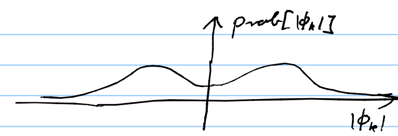
- Assume the system is in a state, $|\alpha\rangle$, other than $|\Psi_0\rangle$.

  $\Rightarrow$ Not for all modes' oscillators is $|\alpha\rangle$ the ground state.

- But if an oscillator is excited, then its wave function spreads out - classically its amplitude of oscillation would increase.

  $\Rightarrow$ If a mode k is excited then the prob. distribution of the ϕ_k spreads:

ground state



example of excited state

- The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a $f_k = \phi_k$ with a large modulus $|\phi_k|$.

  $\Rightarrow$ If, e.g., a mode k is very highly excited then $|\phi_k|$ is likely very large, i.e., a measurement of $\hat{\phi}(x)$ will likely yield a $f(x)$ which shows a plane wave in the direction k with large amplitude - on top of the usual quantum fluctuations.

The particle interpretation

- General states, i.e., states $|\alpha\rangle$ other than the vacuum state $|\Psi_0\rangle$ are states "with particles". Why?
- Recall:

$$\hat{H} = \sum_k \left(\frac{1}{2} \hat{\pi}_k^+(t) \hat{\pi}_k(t) + \frac{1}{2} \hat{\phi}_k^+(t) \left(k^2 + m^2\right) \hat{\phi}_k(t) \right)$$

commuting

$$= \sum_k \hat{H}_k \qquad \text{with } \hat{H}_k = \frac{1}{2} \hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2} \hat{\phi}_k^+ \left(k^2 + m^2\right) \hat{\phi}_k$$

$\Rightarrow$ Any energy eigenstate of the QFT is also eigenstate to each $\hat{H}_k$ – whose spectrum is discrete!

$E_k(n) = \hbar \omega_k \left(\frac{1}{2} + n_k\right)$

$\Rightarrow$ Any energy eigenstate $|E\rangle \in \mathcal{H}$ of the QFT can be specified by listing to which energy level n_k each mode k is excited:

$$|E\rangle = |\{n_k\}_{\text{all } k}\rangle$$

- Example: $|E\rangle = |n_{k_a} = 3, n_{k_b} = 7, \text{ all other } n_k = 0\rangle$

* $|E\rangle$ is the 3rd and 7th excited state for $\hat{H}_{k_a}$ and $\hat{H}_{k_b}$ respectively

* $|E\rangle$ is the ground state for all other $\hat{H}_k$.

□ Energy: using $E_{n_k} = \hbar\omega_k(n_k + \frac{1}{2})$, $\hbar = 1$

$$\hat{H}|E\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2}\omega_k\right)|E\rangle$$

□ Crucial observation:

* If we increase the n_k of a mode k by 1

⇝ total energy increases by $\omega_k = \sqrt{k^2+m^2}$!

* But recall from special relativity: $E^2 - p^2 = m^2$.

⇒ $E_{particle} = \sqrt{k_{particle}^2 + m_{particle}^2} = \omega_k$

⇝ Interpretation (which works at least in Minkowski space:)

$$\boxed{\text{Mode excitation} = \text{particle creation}}$$

□ Example:

If the QFT is, e.g., in the above state $|E\rangle$ then we have 3 and 7 particles of momentum k_a and k_b respectively.

□ Limitations: In general, mode oscillators choice nontrivial!

⇒ This interpretation above is not always applicable!