From Heisenberg to Schrödinger picture

Water:
\[ \phi(x, t) \]

Probe amplitudes, e.g., with a cork:

Quantum field:
\[ \hat{\phi}(x, t) \]

How to visualize an operator-valued field? For now...

Assume we have some means to measure
\[ \hat{\phi}(x, t) \]
at a time \( t \) for all \( x \in \mathbb{R}^3 \).

Q: Why possible in principle?

A: Because \( \hat{\phi}(x, t) = \phi(x, t) \) and \( [\hat{\phi}(x, t), \hat{\phi}(x', t)] = 0 \) \( \forall x, x' \in \mathbb{R}^3 \)

Note: The \( \hat{\phi}(x, t) \) \( \forall x \in \mathbb{R}^3 \) are a maximal set of commuting observables.

\Rightarrow \text{At each } x \text{ we obtain a real-valued measurement outcome, say } f(x).
In vacuum, a typical measurement outcome \( f(x) \) is:

Shown are the level curves.

The measurement collapsed the system into the new state

\[ 1\Phi \in \mathcal{H} \]

which is joint eigenstate of all \( \hat{f}(x,t) \):

\[ \hat{f}(x,t) 1\Phi = f(x) 1\Phi \]

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Here:

If \( f : \mathbb{R}^3 \to \mathbb{R} \) is an arbitrary function, we denote by

\[ 1\Phi \in \mathcal{H} \]

the joint eigenvector of all \( \hat{f}(x,t) \) with eigenvalues \( f(x) \):

\[ \hat{f}(x,t) 1\Phi = f(x) 1\Phi \quad \text{for all } x \in \mathbb{R}^3 \]

Unique up to a phase.

**Hilbert basis:** The set \( \{ 1\Phi \} \)

of all joint eigenvectors of the \( \hat{f}(x,t) \) for all \( x \in \mathbb{R}^3 \) can be used to form a "complete ON basis" of \( \mathcal{H} \). (up to fundamental adjectives).

\[ 1\Phi = \int f(x) 1\Phi \text{ with } \int f(x) dx = 1 \]

Analogous to:

\[ 1\Phi = \int f(x) 1\Phi \text{ with } \int f(x) dx = 1 \]
The "Wave functional"

Recall QM: Assume $\hat{R}_i \psi_i$ is complex set of commuting observables, with joint eigenvectors $1\gamma$ obeying: $\hat{R}_i 1\gamma = \lambda_i 1\gamma$.

Then, the function $\Psi$, given by $\Psi(\tau) = \langle \tau | 1\gamma \rangle$ is called the "wave function" of $1\gamma$ in the $\{|\hat{R}_i \rangle\}$ basis.

Example: $\{p, \hat{p}, \hat{p} \}^2$ yield mom. wave functions $\Psi(p) = \langle p | 1\gamma \rangle$ with $\hat{p} = \hat{p}, \hat{p}, \hat{p}$.

In QFT: e.g., $\{\phi(x)\}_{x \in \mathbb{R}^3}$ is complex set of observables with joint eigenvectors $1\gamma$ obeying $\hat{\phi}(x) 1\gamma = \hat{\phi}(x) 1\gamma$.

Then, $\Psi$, given by $\{1\gamma\}_{\lambda}$ form field on eigen basis

$\Psi[\phi] := \langle \phi | 1\gamma \rangle$ is called the "wave functional".

Alternatively, could use e.g. joint eigenbasis of the $\hat{A}(x, t)$.

Interpretation of $\Psi[\phi]$?

Assume the system is in an arbitrary state $1\gamma \in \mathcal{H}$ at $t$.

If measuring mom. $\hat{\phi}(x, t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $\hat{\phi}(x)$?

Answer: $\text{prob}(|\phi(\tau) \rangle \rightarrow |\phi(\tau) \rangle) = |\langle \phi | 1\gamma \rangle|^2 = |\Psi[\phi]|^2$.
Q: The eqn. of motion for $\Psi[f,t]$?
A: The QFT Schrödinger equation!

- For every quantum theory, we have in the Schrödinger picture of the time evolution:
  \[
  \frac{i}{\hbar} \frac{d}{dt} \Psi(t) = \hat{H} \Psi(t)
  \]
- Which form does it take for $\Psi[f,t]$?
- Here in QFT: now independent of time!
  \[
  \hat{H} = \int \frac{1}{2} \left( \hat{\pi}^2(x) + \hat{\phi}(x)(-\Delta + m^2) \hat{\phi}(x) \right) d^3 x
  \]

But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functions $\Psi[f,t]$?

- A valid representation of $[\hat{\phi}(x), \hat{\pi}(x)] = i\delta'(x-x')$ in:
  \[
  \hat{\phi}(x). \Psi[f,t] = f(x) \Psi[f,t]
  \]
  \[
  \hat{\pi}(x). \Psi[f,t] = -i \frac{\delta}{\delta f(x)} \Psi[f,t]
  \]
- Therefore:
  \[
  \hat{H} = \int \left( \frac{1}{2} \left( -\frac{\delta^2}{\delta f^2(x)} + f(x)(-\Delta + m^2) f(x) \right) \right) d^3 x
  \]
- It is more convenient to use infrared-regularized momentum space:

- Exercise: check
We now need to represent

$$[\hat{p}_k, \hat{n}_k] = i\delta_{k,-k},$$

on the wave functionals $\Psi[\vec{f}, t]$. ($\hat{f}_k$ is Fourier transform of $f(x)$)

As you should verify, this works:

$$\hat{p}_k \cdot \Psi[\vec{f}, t] = \hat{f}_k \Psi[\vec{f}, t]$$

$$\hat{n}_k \cdot \Psi[\vec{f}, t] = -i\frac{\partial}{\partial f_{-k}} \Psi[\vec{f}, t]$$

Note: Ordinary derivatives here because set of variables \( \{ \hat{f}_k \} \) is discrete, since \( k = \frac{2\pi}{\ell} (n_1, n_2, n_3), n \in \mathbb{Z}^3 \).

Schrödinger equation:

$$i\partial_t \Psi[\vec{f}, t] = \sum_k \left( -\frac{\partial}{\partial f_k} \frac{\partial}{\partial f_{-k}} + (k^2 + \omega^2) \hat{f}_k \hat{f}_{-k} \right) \Psi[\vec{f}, t]$$

Recall: For QM harmonic osc., ground state Schrödinger wave function is:

$$\Psi(x, t) = N e^{-\frac{1}{2} \omega x^2 - i\omega t}$$

Exercise: check it. Can you solve for excited states?

Ground state solution in QT reads, similarly:

$$\Psi[\vec{f}, t] = N e^{-\sum_k \left( \frac{1}{2} \omega_k \hat{f}_k \hat{f}_{-k} - i\omega_k t \right)}$$

... which we had already claimed before.
Generic wave functionals

- Assume the system is in a state $|d\rangle$, other than $|\Psi_0\rangle$.

  $\Rightarrow$ Not for all modes' oscillators in $|d\rangle$ the ground state.

- But if an oscillator is excited, then its wave function spreads out - classically its amplitude of oscillation would increase.

  $\Rightarrow$ If a mode $k$ is excited then the prob. distribution of the $\phi_k$ spreads:

![Diagram showing probability distribution changes]

- The more a mode $k$ is excited, the more likely is a measurement of $\hat{P}_k$ to yield a $\phi_k = \phi_x$ with a large modulus $|\phi_x|$.

  $\Rightarrow$ If, e.g., a mode $k$ is very highly excited then $|\phi_k|$ is likely very large, i.e., a measurement of $\hat{\phi}(x)$ will likely yield a $f(x)$ which shows a plane wave in the direction $k$ with large amplitude - on top of the usual quantum fluctuations.
The particle interpretation

- General states, i.e., states $|\phi\rangle$ other than the vacuum state $|\Omega\rangle$ are states "with particles". Why?

- Recall:

\[ \hat{H} = \sum_k \left( \frac{\hat{p}_k^2}{2m} + \frac{\hat{\mathbf{p}}_k^2}{2m} + \frac{1}{2} \phi_k^+(t) \left( k^2 + m^2 \right) \phi_k(t) \right) \]

- \[ \text{comuting} \]

- \[ = \sum_k \hat{H}_k \quad \text{with} \quad \hat{H}_k = \frac{\hat{p}_k^2}{2m} \phi_k + \frac{1}{2} \phi_k^+ \left( k^2 + m^2 \right) \phi_k \]

\[ \Rightarrow \text{Any energy eigenstate of the QFT is also an eigenstate to each } \hat{A}_k \text{ - whose spectrum is discrete!} \]

\[ \mathcal{E}_k(n_k) = \hbar \omega_k \left( \frac{n_k}{2} + 1 \right) \]

\[ \Rightarrow \text{Any energy eigenstate } |E\rangle \in \mathcal{H} \text{ of the QFT can be specified by listing to which energy level } n_k \text{ each mode } k \text{ is excited:} \]

\[ |E\rangle = |E_{n_k}^2 \rangle \]

- Example: \[ |E\rangle = |n_3 = 3, n_7 = 7, \text{ all other } n_k = 0 \rangle \]

\[ \ast |E\rangle \text{ is the 3rd and 7th excited state for } \hat{A}_3 \text{ and } \hat{A}_7 \text{ respectively} \]

\[ \ast |E\rangle \text{ is the ground state for all other } \hat{A}_n \]
Energy: 

\[ E_n = \hbar \omega_k (n_k + \frac{1}{2}) \]

\[ \hat{H} |E\rangle = \left( 3 \omega_k + 7 \omega_b + \sum_{\alpha} \frac{1}{2} \omega_{\alpha} \right) |E\rangle \]

Crucial observation:

* If we increase the \( n_k \) of a mode \( k \) by 1

\[ \Rightarrow \text{total energy increases by } \omega_k = \sqrt{k^2 + m^2} \]

* But recall from special relativity: \( E^2 - p^2 = m^2 \).

\[ \Rightarrow \quad E_{\text{particle}} = \sqrt{k_{\text{particle}}^2 + m_{\text{particle}}^2} = \omega_k \]

Interpretation (which works at least in Minkowski space):

\[ \text{Mode excitation} = \text{particle creation} \]

Example:

If the QFT is, e.g., in the above state \( |E\rangle \) then we have 3 and 7 particles of momentum \( k_a \) and \( k_b \), respectively.

Limitations: In general, mode oscillators choose nontrivial!

\[ \Rightarrow \text{This interpretation above is not always applicable!} \]