The driven harmonic oscillator cont'd:

D. Energy eigenstates

* Recall \( \hat{H}(t) = \omega (a^+(t)a(t) + \frac{1}{2}) - \frac{1}{\sqrt{2\omega}} (a^+(t) + a(t)) J(t) \)

\[
\begin{align*}
\hat{H}(t) &= \begin{cases} 
\omega (a_{\text{in}}^+ a_{\text{in}} + \frac{1}{2}) & \text{for } t < 0 \\
\text{something} & \text{for } 0 \leq t \leq T \\
\omega (a_{\text{out}}^+ a_{\text{out}} + \frac{1}{2}) & \text{for } T < t
\end{cases}
\end{align*}
\]

Here, \( a_{\text{in}} = a(0) \), \( a_{\text{out}} = a(T) \) and \( a_{\text{out}} = a_{\text{in}} + J_0 \).

with: \( J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt' \)

* For \( t < 0 \), we diagonalized the Hamiltonian

\[
\hat{H}(t) = \omega (a_{\text{in}}^+ a_{\text{in}} + \frac{1}{2}) = \hat{H}_{\text{in}} = \text{const.}
\]

by using \( [a_{\text{in}}, a_{\text{in}}^+] = 1 \) to construct its eigenvectors:

\[
\hat{H}_{\text{in}} |n_{\text{in}}\rangle = E_n^{(\text{in})} |n_{\text{in}}\rangle
\]

Namely:

\[
E_n^{(\text{in})} = \omega (n + \frac{1}{2}) , \quad n = 0, 1, 2, 3, \ldots
\]

\[
|n_{\text{in}}\rangle := \frac{1}{\sqrt{n!}} (a_{\text{in}}^+)^n |0_{\text{in}}\rangle
\]

Note: The set \( \{ |n_{\text{in}}\rangle \} \) is a Hilbert basis of the Hilbert space \( \mathcal{H} \).
* By \( t > T \), the Hamiltonian has become a different operator:
\[
\hat{H}(t) = \omega (a_{out}^+ a_{out} + \frac{1}{2}) = \hat{H}_{co,T} = \text{const.}
\]

What are its eigenvectors \( |m_{out}\rangle \) and eigenvalues \( E_m \)?

Observation:

We have:
\[
[a_{out}, a_{out}^+] = 1
\]

\( \Rightarrow \) we can construct the eigenvectors of \( \hat{H}_{co,T} \) with the same method as the eigenvectors of \( \hat{H}_{co} \):

---

* There is a unique vector \( |0_{out}\rangle \in \mathcal{H} \) obeying:
\[
a_{out} |0_{out}\rangle = 0
\]

* We define the set of vectors \( \{ |m_{out}\rangle \} \setminus |0_{out}\rangle \):
\[
|m_{out}\rangle := \frac{1}{\sqrt{m!}} (a_{out}^+)^m |0_{out}\rangle
\]

The operators \( \hat{H}_{co} \) and \( \hat{H}_{co,T} \) are different and have different eigenvectors: \( |m\rangle \) and \( |m_{out}\rangle \). Why are the eigenvalues the same? They both denote a free oscillator of frequency

* Proposition:
\[
\hat{H}_{co,T} |m_{out}\rangle = E_m (m_{out}) |m_{out}\rangle \text{ with } E_m = \omega (m + \frac{1}{2}) = E_m^{(co)}
\]

* Proposition:

The set \( \{ |m_{out}\rangle \} \) is a ON Hilbert basis of the Hilbert space \( \mathcal{H} \).
How are the two bases related?

* Recall: Both, \( |\text{in}\rangle \) and \( |\text{out}\rangle \) are ON bases for \( \mathcal{H} \).

\( \implies \) Each basis' vector \( |\text{in}\rangle \) is a linear combination of the basis' vectors \( |\text{out}\rangle \) and vice versa.

* Therefore, in particular:

There must exist coefficients \( \Lambda_n \in \mathbb{C} \) so that:

\[
|\text{in}\rangle = \sum_n \Lambda_n |\text{out}_n\rangle
\]

\( \overset{\text{"Bogolubov Transformation"}}{\implies} \)

* Meaning of the \( \Lambda_n \):

\( \Delta \) The system is frozen in state \( |\text{in}\rangle = |\text{out}_m\rangle \).

\( \Delta \) Assume we measure at a time \( t > T \) the energy, i.e., we measure

\[
\hat{A}(t) = \omega (a^{+}_\text{out} a^{+}_\text{out} + \frac{1}{2})
\]

\( \Delta \) What is the probability amplitude for finding the energy eigenvalue \( E_n \)?

\( \Delta \) Clearly:

\[
\text{prob. amp. (} |\text{out}\rangle \text{ at } t > T) = \langle \text{out}_n | \text{out}\rangle
\]

i.e.:

\[
\text{prob. (} |\text{out}\rangle \text{ at } t > T) = \left| \langle \text{out}_n | \text{out}\rangle \right|^2
\]
Calculate:

\[ \langle n_{out} | x | n_{in} \rangle = \langle n_{out} | \sum_m \lambda_m | n_{out} \rangle = \lambda_n \]

\[ \Rightarrow \text{If the oscillator started in its ground state, then at time } t > T \text{ the probability for finding the oscillator in its } n \text{-th excited state is given by:} \]

\[ \text{prob.} (|n_{out} \rangle \text{ at } t > T) = |\lambda_n|^2 \]

Remark: In QFT, this will be the prob. for finding \( n \) particles after charges and currents \( J(x,t) \) exited the vacuum.

Calculation of \( \lambda_n \):

**Proposition:** \[ \lambda_n = e^{-\frac{1}{2} |\alpha|^2 \over \hbar \varepsilon_n} \]

**Proof:** The claim is that \[ |n_{out} \rangle = \sum_n e^{-\frac{1}{2} |\alpha|^2 \over \hbar \varepsilon_n} |n_{out} \rangle. \]

We need to check that indeed: \[ a_{in} |0_{in} \rangle = 0 \]

Using \[ a_{out} = a_{in} + J_0 \], we need to check: \[ (a_{out} - J_0) |0_{in} \rangle = 0 \]

Indeed:

\[ (a_{out} - J_0) \sum_n e^{-\frac{1}{2} |\alpha|^2 \over \hbar \varepsilon_n} J_0^n |n_{out} \rangle \overset{\hbar \varepsilon_n}{=} \]

\[ = e^{\frac{1}{2} |\alpha|^2 \over \hbar} (a_{out} - J_0) \sum_n J_0^n \hbar \varepsilon_n \hbar \varepsilon_n (a_{out}^+)^n |0_{out} \rangle \]
\[ e^{-\frac{i}{\hbar}\int_0^t (a_{out} - j_0) e^{J_0 \cdot J_{out}} |_{\text{out}} > \]

\[ = e^{-\frac{i}{\hbar}\int_0^t (a_{out} e^{J_0 \cdot a_{out}} - j_0 e^{J_0 \cdot a_{out}}) |_{\text{out}} > \]

using \[ AB = [A,B] + BA \]

\[ = e^{-\frac{i}{\hbar}\int_0^t \left( [a_{out}, e^{J_0 \cdot a_{out}}] + e^{J_0 \cdot a_{out}} - j_0 e^{J_0 \cdot a_{out}} \right) |_{\text{out}} > \]

\[ = e^{-\frac{i}{\hbar}\int_0^t \left( j_0 - j_0 \right) e^{J_0 \cdot a_{out}} + e^{J_0 \cdot a_{out}} |_{\text{out}} > = 0 \quad \checkmark \]

\textbf{Note:} In the last step, (\text{x}), we used that: \([a_{out}, e^{J_0 \cdot a_{out}}] = j_0 e^{J_0 \cdot a_{out}} \).

\textbf{Exercise:} Show that, more generally, \([a, a^2] = 1 \) implies \([a, f(a^4)] = f'(a^4) \) by induction.

\textbf{Hint:} Show that: \([a, a^4] = 1, [a, a^2] = 2a, [a, a^4] = 3a^2, \ldots, [a, a^{4n-2}] = na^{4n-2} \]

\textbf{Exercise:} Verify that \( |\text{out} > = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}\int_0^t J_{out}} |_{\text{out}} > \) obeys \( \langle \text{out} | \text{out} \rangle = 1 \).

\textbf{Drive mode oscillators in QFT:}

\textbf{Making waves...}

\textbf{Making EM waves...}
Recall:
\[ \hat{H}(t) = \frac{i}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2) \hat{\phi}(x,t) + \hat{J}(x,t) \hat{\phi}(x,t) \, d^3x \]

Example interpretation:

* \( \hat{\phi}(x,t) \) may be viewed as a slightly simplified version of the quantum electromagnetic field.

* \( \hat{J}(x,t) \) may be viewed as a simplified version of a given classical electric charge and current density function.

Example:

A (Klein-Gordon) charge traveling a path \( \tilde{x}(t) \):

Then: \( \hat{J}(x,t) = q \delta(x - \tilde{x}(t)) \)

In- and out periods

Let us consider the case where

\[ \hat{J}(x,t) = 0 \quad \text{for all } t \notin [0,T] \]

\[ \Rightarrow \text{ It suffices to consider the periods } t < 0 \text{ and } t > T \]

in both of which \( \hat{J}(x,t) = 0 \) (and then to relate the bases).

The free (i.e., vacuum) QFT:

\[ \hat{H}(t) = \frac{i}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2) \hat{\phi}(x,t) \, d^3x \]
We need to solve:

\[ \hat{H}(x,t) - (\Delta - m^2) \hat{\phi}(x,t) = 0 \]

\[ \left[ \hat{\phi}(x,t), \hat{\pi}(x',t) \right] = i \delta^3(x - x') \]

* Fourier transformed,

\[ \hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\phi}(x,t) e^{-ik \cdot x} \, dx \]

we need to solve:

\[ \hat{\phi}_k(t) + (k^2 + m^2) \hat{\phi}_k(t) = 0 \quad (EoM) \]

\[ \left[ \hat{\phi}_k(t), \hat{\pi}_k(t) \right] = i \delta^3(k + k') \quad (CCRs) \]

* Recall: \( \hat{\phi}^+_k(t) = \hat{\phi}^*(k,t) \) means \( \hat{\phi}_k(t) = \hat{\phi}^+_k(t) \).

Solution strategy due to Fock:

* Proceed analogously to the driven oscillator, e.g., changing \( \epsilon \leq 0 \):

  - Introduce new variables:

    \text{QM:} \quad a(t) := \sqrt{\frac{\omega}{2}} \hat{\phi}(t) + i \frac{1}{\sqrt{2 \omega}} \hat{\pi}(t)

    \text{QFT:} \quad a_k(t) := \sqrt{\frac{\omega}{2}} \hat{\phi}_k(t) + i \frac{1}{\sqrt{2 \omega}} \hat{\pi}_k(t)

  - Equation of motion and CCRs:

    \text{QM:} \quad \dot{a}(t) = -i \omega a(t) \quad [a(t), a^*(t)] = 1 \quad \text{(verify)}

    \text{QFT:} \quad \dot{a}_k(t) = -i \omega_k a_k(t) \quad [a_k(t), a^*_k(t)] = \delta^3(k - k')

  - Remark: Valid only while no force and \( \omega \) is constant.
Solution, using an initial condition:

QM: \[ a(t) = e^{-i\omega t} a_{in} \], \[ \left[a_{in}, a_{in}^\dagger\right] = 1 \]

QFT: \[ a_k(t) = e^{-i\omega t} a_{in_k} \], \[ \left[a_{in_k}, a_{in_k}^\dagger\right] = \delta^3(\mathbf{k} - \mathbf{k}) \]

\[ \hat{\phi}_k(t) = \frac{1}{\sqrt{V}} \left( e^{-i\omega t} a_{in_k} + e^{i\omega t} a_{in_k}^\dagger \right) \]

\[ \hat{\phi}(x, t) = \sqrt{\frac{\hbar}{2\pi}} \int \frac{dx}{\sqrt{2\omega}} \left( a_{in_x} e^{-i\omega t + ikx} + a_{in_x}^\dagger e^{i\omega t - ikx} \right) \]

\[ \text{Exercise: verify} \]

\[ \int \frac{dx}{\sqrt{2\omega}} \left( a_{in_x} e^{-i\omega t + ikx} + a_{in_x}^\dagger e^{i\omega t - ikx} \right) \]

The Hilbert space of states:

\* Analogous to the QM, there is a vector, \( |0_{in}\rangle \in \mathcal{H} \), which obeys:

\[ a_{in_k} |0_{in}\rangle = 0 \text{, now for all vectors } \mathbf{k} \]

\[ |0_{in}\rangle = \bigotimes_{\mathbf{k}} |0_{in\mathbf{k}}\rangle \]

\* The Hamiltonian reads (for \( t < 0 \)):

\[ \hat{H} = \frac{1}{2} \int_{\mathbb{R}^3} \frac{\hbar^2}{\pi \hbar} \frac{1}{\omega_k} \left( a_{in_k}^\dagger a_{in_k} + \frac{1}{2} \delta^3(0) \right) d^3k \]

In a box:

\[ \hat{H} = \frac{L^{3/2}}{2} \sum_k \omega_k \left( a_{in_k}^\dagger a_{in_k} + \frac{1}{2} \right) \]

(notice: The divergence \( \sum_k L^{3/2} \omega_k \frac{1}{2} = \infty \) is an "ultraviolet divergence."
After the driving ends, $t > T$:

**One obtains:** $a_k(t) = e^{-i\omega k \tau} a_{out_k}$ with $a_{out_k} = a_{in_k} + J_{out_k}$

$$J_{out_k} := \frac{i}{\hbar \omega} \int_0^T J_k(t') e^{i\omega t'} dt'$$

Here: $J_k(t)$ is the Fourier transform of $J(x,t)$.

**Construct the out-basis $|\psi_{out}\rangle \geq \mathbb{F}$ from:**

$$\langle a_{out_k} | 0_{out} \rangle = 0$$

\[ \Rightarrow \text{can calculate, e.g.,} \quad |\langle a_{out_k} | 0_{in} \rangle|^2 \quad \text{i.e., the} \]

probability for $J(x,t)$ to have created $n$ particles of momentum $k$.

---

Recall:

Making waves...

Making EM waves...

Described by $J(x,t)$.
Give the charge $j(x,t)$ its own dynamics:

A code with a spring.

Described by QM, e.g. atom or qubit

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \dot{\phi}(x,t)^2 - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t)\phi(x,t) \, d^3x$$

and upgrade it to:

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} 1 \otimes \left( \dot{\phi}(x,t)^2 - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t)\Phi(x,t) \right) \, d^3x$$

with $j(x,t) = \hat{\phi}(t) \delta(x - \widetilde{x}(t))$

The Hilbert space: $\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{atom}} \otimes \mathcal{H}_{\text{field}}$

Simplified notation:

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \dot{\phi}(x,t)^2 - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t)\Phi(x,t) \, d^3x$$
The charged systems can act as emitters and as receivers of waves.

And, quantumly, they can act as emitters and receivers of particles!

Definition: (Unruh, deWitt):
A "particle", such as a photon, is what a "particle detector", such as an atom, can detect, by getting excited.

With acceleration:

An accelerated atom's charge can excite the field.
This, in turn, can excite the atom: the Unruh effect

⇒ An accelerated atom may detect particles
⇒ If inertial observers only see the vacuum
⇒ Related to gravity via the equivalence principle
⇒ Related to Hawking radiation.