

The Unruh effect (W. G. Unruh, 1976)



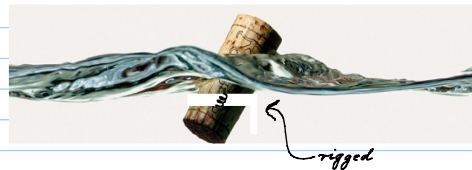
An accelerated ice cube will melt, even in vacuum.

The Unruh effect is the observation, by accelerated observers, of particles, even when the field is in the vacuum state in Minkowski space, i.e., even if inertial observers don't see particles.

Intuition:

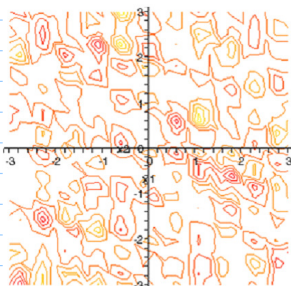


A rigged cork can act as sender and detector.



When accelerated, it gets excited - as if detecting.

Similarly:



An atom (or other Unruh DeWitt detector system) can also both emit and detect: A definition of "particle".



Unruh effect
↓

When accelerated, expect particle emission and detection.

We'll consider detectors at rest and in motion:

* A detector at rest has: $x^\mu(\tau) = (\tau, 0, 0, 0)$

* Case of constant velocity:

$$x^\mu(\tau) = (a\tau, \vec{b}\tau)$$

with $a^2 - \vec{b}^2 = 1$. Exercise: verify

* Case of constant acceleration in the x -direction:

$$x^0(\tau) = d \sinh(\tau/d)$$

$$x^1(\tau) = d (1 + \sinh^2(\tau/d))^{1/2}$$

$$x^2(\tau) = x^3(\tau) = 0$$

Exercise: \square verify that $\ddot{x}^\mu \ddot{x}_\mu = \text{const}$
(i.e. for still small velocities)
 \square show that for $\tau \ll 1$: $x(\tau) \approx (\tau, a + b\tau^2)$

The quantum field

\square We assume that, for an inertial observer, the field $\hat{\phi}$ is in the Minkowski vacuum. Recall:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ikx} \hat{\phi}_k(x^0) d^3k \quad \text{with} \quad \hat{\phi}_k(x^0) = \frac{1}{\sqrt{2}} \left(v_k^+(x^0) a_k + v_k^-(x^0) a_k^\dagger \right)$$

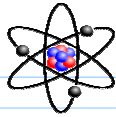
$$\text{and} \quad v_k^\pm(x^0) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0} \quad \text{with} \quad \omega_k = \sqrt{k^2 + m^2}.$$

$$\square \text{ Thus: } \hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 + i\vec{k}\vec{x}} a_k + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}} a_k^\dagger \right) d^3k$$

\square Note: $\hat{\phi}(x)$ acts on Hilbert space $\mathcal{H}^{\text{field}}$.

Next: Consider a system (e.g. an atom) that can detect particles of the Klein Gordon field $\hat{\phi}$.

The detector system



- Small, localized system with path $x^\mu(\tau)$
 E.g.: * An atom
 * An oscillator, such as the diatomic molecule H_2 .

Inertial observer's
cartesian coordinates.
↓
↑
detector's eigen time

- First quantized.
- Hamiltonian $\hat{H}^{\text{detector}}$ acts on Hilbert space $\mathcal{H}^{\text{detector}}$.
- Assume $\text{spec}(\hat{H}^{\text{detector}}) = \{E_0, E_1, E_2, \dots\}$ is discrete.

⇒ The total quantum system thus consists of two subsystems, with:

$$\hat{H}^{\text{total}} = \hat{H}_0^{\text{detector}} \otimes 1 + 1 \otimes \hat{H}_0^{\text{field}} + \hat{H}^{\text{interaction}}$$

- On the total Hilbert space:

$$\mathcal{H}^{\text{total}} = \mathcal{H}^{\text{detector}} \otimes \mathcal{H}^{\text{field}}$$

- The interaction Hamiltonian $\hat{H}^{\text{interaction}}$ consists of operators of both subsystems, usually:

$$\hat{H}^{\text{interaction}}(\tau) = \epsilon(\tau) \hat{Q}(\tau) \hat{\phi}(x^0(\tau), \vec{x}(\tau))$$

Detector efficiency
(can also be used
as on/off switch)

An observable
of the detector's
quantum system

The field $\hat{\phi}$
at the current
detector location

- Examples: $\hat{H}^{\text{int}} = \hat{S}_z(\tau) \otimes \hat{B}_z(x(\tau))$

↑
detector is a spin.

↑
field is magnetic field.

or: $\hat{H}^{\text{int}} = -\frac{e}{mc} \hat{p}_i \otimes \hat{A}^i(x(\tau))$

(use Axial gauge: $\partial_i A^i = 0$)

Time evolution

- If we (realistically) assume that the detector efficiency $\varepsilon(\tau)$ is small, we can use perturbation theory.
- In this case, the Dirac picture of time evolution is convenient:

* Operators evolve according to

$$\hat{H}^{\text{free}} = \hat{H}^{\text{detector}} \otimes 1 + 1 \otimes \hat{H}^{\text{field}} \quad (*)$$

For example:

$$\begin{aligned} \hat{Q}(\tau) &= e^{i\hat{H}^{\text{free}}\tau} (\hat{Q}_0 \otimes 1) e^{-i\hat{H}^{\text{free}}\tau} \\ &= e^{i\hat{H}^{\text{detector}}\tau} \hat{Q}_0 e^{-i\hat{H}^{\text{detector}}\tau} \otimes 1 \end{aligned}$$

* States evolve according to $\hat{H}^{\text{int}}(\tau)$, i.e., according to the time evolution operator:

$$\hat{U}(\tau) = T \exp \left(i \int_{\tau_i}^{\tau_f} \hat{H}^{\text{interaction}}(\tau') d\tau' \right)$$

↑
time-ordering symbol

↑
In $\hat{H}^{\text{interaction}}$ the operators are time dependent, evolving according to (*)

Perturbative ansatz

- For small detector efficiency $\varepsilon(\tau)$ we can expand:

$$\hat{U}(\tau) = 1 + i \int_{-\infty}^{\tau} \varepsilon(\tau') \hat{Q}(\tau') \hat{\phi}(x(\tau'), \bar{x}(\tau')) d\tau' + \mathcal{O}(\varepsilon^2)$$

- Note: We can set $\tau_i = -\infty$ since we can always switch $\varepsilon(\tau)$ on or off.

Initial conditions

- We assume that the quantum field $\hat{\phi}$ starts out in a state $|\alpha\rangle$ with $|\alpha\rangle = \text{Minkowski vacuum}$, $|\alpha\rangle = |0\rangle$, or a 1-particle state $|\alpha\rangle = |1_k\rangle$.
- We assume that the detector starts out in its ground state $|E_0\rangle$.
- Thus, the combined system starts out in the state:

$$|\Psi_{in}\rangle = |E_0\rangle \otimes |\alpha\rangle$$

- Time evolution:

At time τ the total system is in the state

$$|\Psi(\tau)\rangle = \hat{U}(\tau) |\Psi_{in}\rangle$$

Particle creation

- The problem:

What is the probability amplitude that, if we measure at time τ the detector system will be found to have detected something, i.e., to be in an excited state $|E_n\rangle$?

- To this end, calculate:

$$p(\tau) := \left(\langle E_n | \otimes \langle \Omega | \right) |\Psi(\tau)\rangle$$

for an arbitrary end state $|\Omega\rangle$ of the quantum field $\hat{\phi}$.

- Note: We will see that not all states $|\Omega\rangle$ yield a nonzero $p(\tau)$.

Total detection probability:

□ The probability for detection eventually is:

$$p(\infty) \approx \langle E_n | \otimes \langle \Omega | \left(1 + i \int_{-\infty}^{+\infty} \varepsilon(\tau) \hat{Q}(\tau) \otimes \hat{\phi}(x(\tau)) d\tau \right) | E_0 \rangle \otimes | \Omega \rangle$$

(we may choose $\varepsilon(\tau)$ so as to make it finite)

Note: $\langle E_n | E_0 \rangle = 0 \Rightarrow 1^{\text{st}}$ term vanishes \Rightarrow

$$= i \int_{-\infty}^{+\infty} \varepsilon(\tau) \langle E_n | \hat{Q}(\tau) | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \Omega \rangle d\tau$$

Recall: $\hat{Q}(\tau) = e^{iH_0 \tau} \hat{Q}_0 e^{-iH_0 \tau}$

$$p(\infty) = i \int_{-\infty}^{+\infty} \varepsilon(\tau) e^{i(E_n - E_0)\tau} \langle E_n | \hat{Q}_0 | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \Omega \rangle d\tau$$

Recall: $\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{2\omega_k}} e^{i\omega_k x^0 - i\vec{k}\cdot\vec{x}} \frac{1}{a_{\vec{k}}} + \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\cdot\vec{x}} \frac{1}{a_{-\vec{k}}} \right) d^3k$

Note: We can now calculate all absorption and emission processes.

Here: Let's focus on particle detection in the vacuum, $|\Omega\rangle := |0\rangle$:

* In $\hat{\phi}(x)$, only the terms $\sim a_{\vec{k}}^+$ can contribute, because $a_{\vec{k}} |0\rangle = 0$

* Thus, in $|\Omega\rangle$ only the one-particle components contribute:

$$|\Omega\rangle = \Omega_0 |0\rangle + \int \Omega_{\vec{k}} a_{\vec{k}}^+ |0\rangle d^3k + \int \int \Omega_{\vec{k}_1, \vec{k}_2} a_{\vec{k}_1}^+ a_{\vec{k}_2}^+ |0\rangle d^3k_1 d^3k_2 + \dots$$

* Thus, let us consider a $|\Omega\rangle = a_{\vec{k}}^+ |0\rangle$:

$$\Rightarrow p(\infty) = i \frac{\langle E_n | \hat{Q}_1 | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \varepsilon(\tau) e^{i(E_n - E_0)\tau} \left\langle 0 | a_{\vec{k}} \int \frac{e^{i\omega_{\vec{k}} x_0(\tau) - i\vec{k}\vec{x}(\tau)}}{\sqrt{2\omega_{\vec{k}}}} a_{\vec{k}}^\dagger d^3k | 0 \right\rangle d\tau$$

leads to:

$$\langle 0 | a_{\vec{k}} a_{\vec{k}}^\dagger | 0 \rangle = \langle 0 | a_{\vec{k}}^\dagger a_{\vec{k}} + \delta^3(\vec{k}-\vec{k}) | 0 \rangle = \delta^3(\vec{k}-\vec{k})$$

\Rightarrow

$$p(\infty) = i \frac{\langle E_n | \hat{Q}_1 | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{e^{i(E_n - E_0)\tau} e^{i(\omega_{\vec{k}} x_0(\tau) - \vec{k}\vec{x}(\tau))}}{\sqrt{2\omega_{\vec{k}}}} \varepsilon(\tau) d\tau$$

some constant

Special case: $|d\rangle = |0\rangle$ and detector inertial:

* Choose the detector's rest frame: $x^\mu(\tau) = (\tau, 0, 0, 0)$

* Thus:

$$p(\infty) = i \frac{\langle E_n | \hat{Q}_1 | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{e^{i(E_n - E_0)\tau} e^{i(\omega_{\vec{k}} x_0(\tau) - \vec{k}\vec{x}(\tau))}}{\sqrt{2\omega_{\vec{k}}}} \varepsilon(\tau) d\tau$$

assume $\varepsilon(\tau) = 1$, i.e., "always on".

$$= i \frac{\langle E_n | \hat{Q}_1 | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{e^{i(E_n - E_0)\tau} e^{i\omega_{\vec{k}}\tau}}{\sqrt{2\omega_{\vec{k}}}} d\tau$$

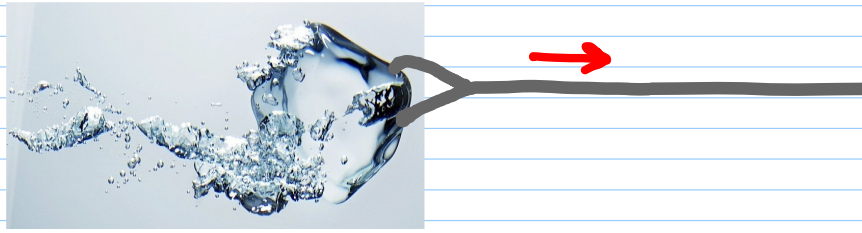
$$= i \frac{\langle E_n | \hat{Q}_1 | E_0 \rangle}{(2\pi)^{3/2}} (2\pi)^{1/2} \int \underbrace{(\overbrace{E_n - E_0 + \omega_{\vec{k}}}^{> 0})}_{\substack{\parallel \\ \sqrt{k^2 + m^2} > 0}} \frac{1}{\sqrt{2\omega_{\vec{k}}}}$$

this cannot be 0

$$= 0$$

\Rightarrow No excitation of the detector, as expected.

Special case: $|d\rangle = |0\rangle$ and detector non-inertial:



□ The probability amplitude for the detector to become excited will depend on the excitation energy $E_{ex} := E_n - E_0$:

$$p(\infty) = \underbrace{i \frac{\langle E_n | \hat{Q}_i | E_0 \rangle}{(2\pi)^{3/2} \sqrt{2\omega_i}}}_{\text{a constant}} \int_{-\infty}^{+\infty} \underbrace{e^{i(E_n - E_0)\tau}}_{\substack{\text{Fourier factor} \\ \text{i.e. } \tau \text{ and } E_{ex} \\ \text{are a Fourier pair} \\ \text{(if neglecting the "constant")}}} \underbrace{e^{i(\omega_n x^0(\tau) - \tilde{k} \tilde{x}(\tau))}}_{\text{function that is being Fourier transformed}} \varepsilon(\tau) d\tau$$

□ Clearly: For generic, accelerated detectors the function

$$f(\tau) := e^{i(\omega_n x^0(\tau) - \tilde{k} \tilde{x}(\tau))} \varepsilon(\tau)$$

possesses a Fourier transform

$$F(E_x) = \int_{-\infty}^{+\infty} e^{iE_x \tau} f(\tau) d\tau, \quad E_x = E_n - E_0$$

which is generally non zero for positive E_x .

$\Rightarrow p(\infty) \sim F(E_x) \neq 0 \Rightarrow$ detector does get excited.
(“proportional to” (European notation) (while also the field gets excited))

\rightsquigarrow Unruh effect

Example: The constantly accelerated detector.

- * One finds that the prob. of excitation is identical to the case in which the detector is in a heat bath of temperature $T \sim \frac{\hbar a}{2\pi c}$ where a is the acceleration.

Assigned reading: Birrell & Davies: p 52-58.

Remark: * Note that both the detector and the quantum field become excited. Is energy conservation violated?

- * One can show that the energy stems from the accelerating agent.

E.g.: Think of a regular antenna. If the accelerated e^- were excitable little systems, they would get excited.

- * It's the case of a system with charge in time-dependent interaction with the field: An antenna where field & system get excited.

Special case: $|d\rangle = |1_k\rangle$:

Recall:

$$P = i \int_{-\infty}^{+\infty} \frac{\varepsilon(\tau)}{\sqrt{2\omega_k}} e^{i(E_n - E_0)\tau} \langle E_n | \hat{Q}_0 | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | d \rangle d\tau$$

Prob. amplitude for detector to get excited

Recall: $\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 - i\vec{k}\cdot\vec{x}} a_{\vec{k}}^\dagger + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\cdot\vec{x}} a_{\vec{k}} \right) d^3k$

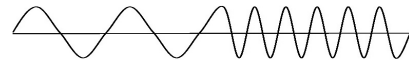
\Rightarrow For $|d\rangle = |1_k\rangle = a_{\vec{k}}^\dagger |0\rangle$, we can have:

a.) $|\Omega\rangle = |2_k\rangle$, or $|1_k, 1_{\vec{k}}\rangle$ Would mean detector excites the field further

b.) $|\Omega\rangle = |0\rangle$: Means detector absorbs a particle.

↙ i.e., not only "detects" a particle.

Alternative intuition



- ▣ A monochromatic wave in an inertial frame is not monochromatic for an accelerated observer.
- ▣ Thus, the accelerated observer's modes are coupled oscillators: he sees wavelengths change.
- ▣ These oscillator's ground state is different.

⇒ Calculations strategy:

- ▣ Use accelerated observer's mode decomposition.
- ▣ Relate it to inertial observer's mode decomposition.
- ▣ Choose vacuum for the inertial observer
- ▣ Calculate particle production for accelerated observer analogous to $|u_{in}\rangle$ to $|u_{out}\rangle$ transform for driven harmonic oscillators' evolution above.