Recall:

- Using different choices of mode functions, $\tilde{V}_b(\eta)$, $\tilde{V}_a(\eta)$, we can write $\tilde{X}_a(\eta)$ in different ways:

\[
\tilde{X}_a(\eta) = \frac{1}{\sqrt{2}} \left( v_k^+ (\eta) a_k + v_k^- (\eta) a_k^- \right) \quad (A)
\]

\[
= \frac{i}{\sqrt{2}} \left( \tilde{v}_k^+ (\eta) \tilde{a}_k + \tilde{v}_k^- (\eta) \tilde{a}_k^- \right)
\]

- Since for each $k$, the space of possible mode functions is 2-dimensional, there exist complex $d_k, \phi_k$ so that:

\[
\tilde{V}_b(\eta) = d_k V_b(\eta) + \phi_k V_b^+(\eta) \quad (B)
\]

(Recall: Because $\tilde{V}_b(\eta)$ must obey the Wronskian condition, $d_k$ and $\phi_k$ must obey $d_k^2 + \phi_k^2 = 1$)

- From (A) and (B) we obtain (exercise):

\[
a_k = d_k^+ \tilde{a}_k + \phi_k \tilde{a}_k^-
\]

- Thus, $a_k |0\rangle = 0$ becomes $(d_k^+ \tilde{a}_k + \phi_k \tilde{a}_k^-) |0\rangle = 0$, which yields:

\[
|0\rangle = \left[ \prod_k \frac{1}{|d_k^+|^2} e^{-\frac{\phi_k^2}{2|d_k^+|^2}} \tilde{a}_k^+ \tilde{a}_k^- \right] |0\rangle \quad (T)
\]

⇒ We can now express all basis vectors $10\rangle$, $a_k^+ |0\rangle$, $a_k^- |0\rangle$, $10\rangle$... in terms of the basis vectors $10\rangle$, $\tilde{a}_k^+ |0\rangle$, $\tilde{a}_k^- |0\rangle$, $10\rangle$...

Example scenario:

* Assume $V_b(\eta)$, $\tilde{V}_b(\eta)$ chosen so that $10\rangle$, $10\rangle$ are vacuum at $\eta_1$, $\eta_2$.
* Assume system is in vacuum state at $\eta_1$, i.e., $10\rangle = 10\rangle$.
* Then system's state $10\rangle$ at $\eta_2$ is an excited state, i.e., a state with particles!
The extent of particle creation?

- Eqn. (T) shows that there is a finite probability amplitude for finding arbitrarily many particles at time $t_e$. Does that mean $\infty$ many get created (at $\infty$ energy expense and thus halting the expansion?)

- Let us calculate the expected number of created particles:

  * Definition (QM): $\hat{N} := a^+ a$ is called a "Number operator".

  * Why? It is a self-adjoint observable with eigenbasis:
    
    $\hat{N}(a^+)^n |0\rangle = n(a^+)^n |0\rangle$

  * Exercise: verify.

- Definition (QFT): $\hat{N}_k := a^+_k a_k$

Interpretation of $\hat{N}_k$ in QFT

- Assume that at some time, $\tau$, the state $|0\rangle$ is the vacuum.

- Thus, at $\tau$, for example the state $(a^+_k)^n |0\rangle$ is a state with $n$ particles of momentum $k$.

- Now assume that at $\tau$ the system is in an arbitrary state $|\omega\rangle$.

- Then, at $\tau$, the expected number of particles of momentum $k$ is:

  $$\overline{N}_k = \langle \omega | \hat{N}_k |\omega\rangle$$
Calculation in the above scenario for $\vec{N}_k := \hat{a}_k^\dagger \hat{a}_k$ at time $q_2$

$$\vec{N}_k = <\Omega | \hat{N}_k | \Omega>$$

$$= <0 | \hat{a}_k^\dagger \hat{a}_k | 10>$$

Now use that $a_k = \hat{a}_k^\dagger \alpha_k + \beta_k \hat{a}_k$, i.e.

also, that $\tilde{a}_k = \hat{a}_k^\dagger a_k + \hat{a}_k \alpha_k$

Exercise: calculate $\vec{N}_k \tilde{a}_k$ in terms of $\alpha_k, \beta_k$.

$$= <0 | (\hat{a}_k a_k^\dagger + \beta_k^* a_k) (\hat{a}_k^* a_k^\dagger + \tilde{a}_k^* \alpha_k^\dagger) | 10>$$

$$= <0 | \beta_k^* \hat{a}_k^\dagger a_k a_k^\dagger + \hat{a}_k a_k^\dagger \alpha_k^\dagger \alpha_k^\dagger | 10>$$

$$= \tilde{\beta}_k \tilde{\beta}_k <0 | a_k^\dagger a_k + 1 | 10>$$

using infrared regularization we have $\tilde{\alpha}_k \tilde{\alpha}_k = \Delta_k^2$ (see Exercise)

$$= \tilde{\beta}_k \tilde{\beta}_k$$

Total particle number:

- The expected total number of particles at time $q_2$ is then:

$$\vec{N} = \sum_k <\Omega | \hat{N}_k | \Omega> = \sum_k \tilde{\beta}_k \tilde{\beta}_k$$

Note:

* We assumed here an infrared, i.e., a box regularization. (Else the number of created particles can only be zero)

* Else, $\vec{N}$ may come out infinite, but that can be ok.

* This happens even for photon creation through moving charges.

* But we always need of course finite “energy”:

$$<\Omega | \hat{N}(q) | \Omega> < \infty$$
Identification of the vacuum state

How can we identify, at any arbitrary fixed time, \( \gamma \), that Hilbert space vector, say \(|\text{vacuum at } \gamma \rangle\), which describes the vacuum, i.e., the no particle state, at that time, \( \gamma \)?

**Q:** Is \(|\text{vacuum at } \gamma \rangle\) one of the (infinitely many) states \( |0\rangle, |1\rangle, |\tilde{1}\rangle, \ldots \) that come with choices of mode functions \( \nu_k, \tilde{\nu}_k, \tilde{\tilde{\nu}}_k, \ldots \) through \( a_k |0\rangle = 0, \tilde{a}_k |0\rangle = 0, \tilde{\tilde{a}}_k |0\rangle = 0, \ldots \)?

**A:** As we will see:

Yes, if \( \gamma \) when \(|\text{vacuum at } \gamma \rangle\) exists at all, then there exist suitable mode functions, \( \nu_k \), (namely exactly one, up to a phase, for each \( k \)) so that with

\[
\hat{x}_k = \frac{i}{\sqrt{2 \hbar}} (\nu_k^* a_k + \nu_k a_{-k}^*)
\]

the state \(|0\rangle\) defined through \( a_k |0\rangle = 0 \) is the vacuum state at the time \( \gamma \):

\[
|\text{vacuum at } \gamma \rangle = |0\rangle
\]
But how to specify \(|\text{vacuum}_\eta\rangle\)?

We notice: To specify \(|\text{vacuum}_\eta\rangle\) by specifying a suitable vector \(|0\rangle\)

is equivalent to

specifying a suitable mode function \(V_\alpha\) (i.e. a suitable solution to the K.G. and Wronskian equations)

is equivalent to

specifying at time \(\eta\) that \(V_\alpha(\eta) = \tau_\kappa, \ V_\alpha'(\eta) = s_\kappa\)

for a suitable choice of \(\tau_\kappa, s_\kappa \in \mathbb{C}\).

(because with the K.G. equation being 2nd order in time, these two conditions suffice to determine the full \(V_\alpha\) at all time)

1st attempt:

- Ansatz:

Let us try to define the vacuum state at a time \(\eta\) as that Hilbert space vector (up to a phase) which at time \(\eta\) minimizes the Hamiltonian, \(H^{(\alpha)}(\eta)\).

- To this end, we will choose \(\tau_\kappa, s_\kappa \in \mathbb{C}\) suitably, so that \(V_\alpha(\eta) = \tau_\kappa, \ V_\alpha'(\eta) = s_\kappa\) define that mode function \(V_\kappa\) so that \(|0\rangle\) is the lowest energy state.
Calculation of the lowest energy state at some arbitrary fixed time, \( \eta \).

\[
\langle 0 | \hat{H}^{(x)}(\eta) | 0 \rangle = \langle 0 | \frac{1}{2} \int_{\text{box}} \hat{\mathcal{H}}^{2}(\eta,x) + \sum_{i,j} \hat{\mathcal{H}}_{ij}^{2}(\eta) \ dx \ dx' | 0 \rangle
\]

\[
+ \left( m^2 a^2(\eta) - \frac{a^2(\eta)}{a(\eta)} \right) \mathcal{H}^{2}(\eta,x) \ dx \ dx' | 0 \rangle
\]

Exercise:

Use Fourier and use

\[
\mathcal{H}_{n}^{2}(\eta) = \frac{1}{N^2} (v_{n}(\eta) a_{n} + v_{n}(\eta) a_{n}^\dagger)
\]

to evaluate this energy expectation value.

Result:

\[
\langle 0 | \hat{H}^{(x)}(\eta) | 0 \rangle = \langle 0 | \frac{1}{4} \sum_{n} \left( v_{n}^{2}(\eta) + \omega_{n}^{2}(\eta) v_{n}^{2}(\eta) \right) a_{n} a_{n}^\dagger
\]

\[
+ \frac{1}{4} \sum_{n} \left( v_{n}^{2}(\eta) + \omega_{n}^{2}(\eta) v_{n}^{2}(\eta) \right) a_{n} a_{n}^\dagger
\]

\[
+ \frac{1}{2} \sum_{n} \left( |v_{n}(\eta)|^2 + \omega_{n}^{2}(\eta) |v_{n}(\eta)|^2 \right)(a_{n} a_{n}^\dagger + |a_{n} a_{n}^\dagger|)
\]

\[
= \frac{1}{4} \sum_{n} |v_{n}(\eta)|^2 + \omega_{n}^{2}(\eta) |v_{n}(\eta)|^2
\]

Here: the time-dependent frequency reads: \( \omega_{n}(\eta) := \frac{1}{4} + m^2 a^2(\eta) - \frac{a^2(\eta)}{a(\eta)} \)

Note: we assume \( \omega_{n}^{2}(\eta) > 0 \) because, else, the potential is inversed and there is no lowest energy state.
Recall:

* We defined \( r_k := V_k(q_i) \), \( s_k := V'_k(q_i) \)
* We need to determine \( r_k, s_k \in \mathbb{C} \)
* This will determine a full mode function \( V_k \) with its \( a_k \)
* This determines a corresponding \( 10 \) obeying \( a_k \langle 10 \rangle = 0 \)
* Our ansatz is then that:
  \[ |\text{vacuum at } q_i \rangle = 10 \]

Concretely:

* From above, the energy at \( q_i \) is:
  \[
  \langle 0 | \hat{H}^{(y)}(q_i) | 10 \rangle = \frac{i}{4} \sum_k | V'_k(q_i) |^2 + \omega^2_k(q_i) | V_k(q_i) |^2
  \]

* Using the definitions \( r_k = V_k(q_i) \), \( s_k = V'_k(q_i) \):
  \[
  \langle 0 | \hat{H}^{(y)}(q_i) | 10 \rangle = \frac{i}{4} \sum_k s_k^* s_k + \omega^2_k(q_i) r_k r^*_k \quad (E)
  \]

* We want to minimize this expression, subject to the Wronskian condition:
  \[ V'_k(q_i) V^*_k(q_i) - V_k(q_i) V'_k(q_i) = 2i \]
  i.e., subject to the constraint:
  \[ s_k r^*_k - r_k s^*_k = 2i \quad (C) \]

* Use Lagrange multiplier \( \lambda \) and extremize:
  \[
  S'(s_k, r_k) := s_k s^*_k + \omega^2_k r_k r^*_k + \lambda (s_k r^*_k - r_k s^*_k)
  \]
We have to solve:

\[ \frac{\partial S}{\partial S_k} = 0 \quad i.e., \quad S_k - \lambda v_k = 0 \]

\[ \frac{\partial S'}{\partial v_k} = 0 \quad i.e., \quad \omega_k^2 v_k + \lambda S_k = 0 \]

along with the constraint (C): \( S_k v_k^+ - v_k S_k^+ = 0 \).

Exercise:

Show that the solution is:

\[ v_k = \frac{1}{\nu \omega_k^2} e^{i\theta} \quad S_k = i \omega_k v_k \quad e^{i\theta} \]

where \( \theta \in [0, 2\pi) \) is arbitrary. We'll choose \( \theta = 0 \).

These conditions at time \( \xi_r \):

\[ v_k(\xi_r) \quad \text{and} \quad v_k'(\xi_r) = i \omega_k v_k(\xi_r) \]

Define a mode function \( v_k \) for all \( \xi \) so that

\[ x^k(\xi_r) = \frac{1}{\nu^2} (v^*_k(\xi_r) a_k + v_k(\xi_r) a_k^+) \]

and the corresponding state \( |0\rangle \) obeying \( a_k |0\rangle = 0 \) is the lowest energy state of the Hamiltonian \( H^m(\xi_r) \),

i.e., the instantaneous lowest energy state at time \( \xi_r \).
Special case: Minkowski space

- Minkowski space is the special case $a(y) = 1$ for all $y$. Then, $\omega^2(y) = b^2 + m^2$ is a constant. Also: $y = t$.

- We conclude that $|0\rangle$ is the state of lowest energy at a time $y$, if we choose the mode functions which obey these conditions:

\[ v_a(y) = \frac{1}{\sqrt{\omega_a}}, \quad v_b(y) = i \sqrt{\omega_a} \]

- Solving the K.B. eqn, we find that these mode functions are:

\[ v_a(y) = \frac{1}{\sqrt{\omega_a}} e^{i(y-y_0)\omega_a} = \frac{1}{\sqrt{\omega_b}} e^{i(t-t_0)\omega_b} \]

Exercise:

* Verify that the state $|0\rangle$ that we have found for Minkowski space agrees with the state that we identified as the Minkowski space vacuum at the beginning of the course.

* Show that, if we, similarly, determine the lowest energy state at another time, $y_2$, then we obtain the same mode function $v_b$ (up to an irrelevant phase).

* This means that the same vector $|0\rangle$ minimizes the energy at all times, on Minkowski space, (which had to come out because of time translation symmetry).
Back to our ansatz, namely the assumption:

At an arbitrary time $\gamma$, the vacuum (no particle) state $|0\rangle$ is the state which is the lowest energy state $|0\rangle$ at time $\gamma$:

$|\text{Vacuum at } \gamma,\rangle = |0\rangle$

- **Implicated prediction:**
  
  Universe expands $\Rightarrow \hat{A}(0, \gamma) + \hat{A}^{\dagger}(0, \gamma)$
  
  $\Rightarrow$ expect particle production, in general.

- **Concretely:** current production rate $\approx 10$ particles $(\text{Km}^3 \text{year} \cdot \text{species})^{-1}$

Experiment: That's much too high! We only have $\approx 10^4$ particles $(\text{Km}^3)^{-1}$

Reconsider:

- Recall that any quantum system does not get excited (or only very little), if we change its parameters (e.g., the $w_0(0)$) slowly.

- For the oscillator, "slow", is slow compared to the natural frequency of the oscillator.

  Only changes in length which occur fast compared to the oscillator's frequency can parametrically excite the oscillator.

- Since the universe presently expands slowly, we should expect essentially no particle production, and indeed we don't see any, experimentally.

  How to improve our ansatz for vacuum identification?
Preliminary consideration

- Consider models where the universe is initially Minkowskian, and then undergoes an expansion whose parameter change (of \( w_0(q) \)) is slow, i.e., adiabatic.

\[ \text{Note: the overall change may still be large!} \]

\[ \Rightarrow \text{ We expect essentially no particle creation.} \]

\[ \Rightarrow \text{ The vacuum state (i.e., no particle state) should always be essentially the same Hilbert space vector.} \]

\[ \Rightarrow \text{ Since there is only one vacuum state, } 1 \mathbf{0}, \text{ for all time, there is one mode function, } V_k, \text{ whose } 1 \mathbf{0} \text{ is the vacuum at all time.} \]

How can we find this mode function \( V_k \)?

- Easy: We know \( V_k(q) \) at very early times, when the universe was still Minkowskian:

\[ V_k(q) = \frac{1}{\sqrt{|w_0(q)|}} e^{i\int_{q_0}^q w_0(q') dq'} \]

\[ \text{\textsuperscript{arbitrary reference time}} \]

Then: the K.G. eqn. yields \( V_k(q) \) at all time!

- Proposition:

\[ V_k(q) = \frac{1}{\sqrt{|w_0(q)|}} e^{i\int_{q_0}^q w_0(q') dq'} \quad \text{(5)} \]

\[ \text{is a very good approximation, if the evolution is "adiabatic"} \]
**Definition:**

We say that a mode $k$ evolves adiabatically slow, if:

\[
\frac{\omega_k'(\gamma)}{\omega_k^2(\gamma)} \ll 1 \quad \text{and} \quad \frac{\omega_k''(\gamma)}{\omega_k^2(\gamma)} \ll 1 \quad (AC)
\]

**Note:**

The denominators are chosen so that the quotients are unitless, because only pure numbers can reasonably be said to be small or large.

**Exercise:** Prove the proposition.

Hint: Show that (5) obeys the K.G. eqn provided the adiabaticity, (AC), holds.

---

Is initial Minkowski period really necessary?

* Try to identify the $v_k$ whose $10^{-7}$ is the adiabatically defined vacuum without referring to what $v_k$ would look like in an earlier Minkowski period of the universe.

* Namely, try to identify $v_k$ by a characteristic property that it has at all times.

* Indeed, we notice: \((Exercise: \ check \ this)\)

Our $v_k$ of (5) above satisfies at all times:

\[
v_k(\gamma) = e^{i \frac{\gamma}{\omega(\gamma)}} , \quad v_k'(\gamma) = (i \omega_k(\gamma) - \frac{i}{2} \frac{\omega_k'(\gamma)}{\omega_k^2(\gamma)}) e^{i \frac{\gamma}{\omega(\gamma)}} \quad (AV)
\]
"The general adiabatic vacuum identification"

Definition:

* Consider an arbitrary time $\tau_1$.

* Assume that the evolution of $\psi_\alpha$ is adiabatically slow for mode $k$, at time $\tau_1$.

* We then identify that state as the vacuum $|0\rangle$ (i.e., the no-particle state) at $\tau_1$, whose mode function $v_k$ is specified by the conditions (AV) at $\tau_1$:

$$v_k(\tau_1) = e^{i\theta} \frac{1}{\sqrt{\omega_\alpha(\tau_1)}}, \quad v'_k(\tau_1) = \left(i \omega_\alpha(\tau_1) - \frac{1}{2} \omega'_\alpha(\tau_1) \right) e^{i\theta}$$

(AV)

* We call this $|0\rangle$ the "adiabatic vacuum" at $\tau_1$.

Remarks:

口 Recall that the criteria for choosing $v_k$ so that its $|0\rangle$ is the lowest energy vacuum at time $\tau_1$, are:

$$v_k(\tau_1) = \frac{1}{\sqrt{\omega_\alpha(\tau_1)}} e^{i\theta}, \quad v'_k(\tau_1) = i \frac{1}{\sqrt{\omega_\alpha(\tau_1)}} e^{i\theta}$$

(EV)

口 Note that AV and EV generally differ!

⇒ The adiabatically-defined vacuum is generally not the lowest energy state!

口 Note that the adiabatic vacuum criterion should only be applied when the evolution of the mode under consideration is actually adiabatic.

口 No vacuum criterion for generic spacetimes is known.