From the particle picture to the wave picture

So far: Spacetime dynamics can produce particles.

When? When mode oscillators \( w_n(t) \) changes nonadiabatically fast: \( w_n(t)/w_n(t)^2 \gg 1 \).

In cosmology? No: particles mostly produced conventionally from inflation potential at the end of inflation.

Now: Spacetime dynamics can enhance quantum field fluctuations!

When? When \( w_n(z) \) becomes imaginary.

Recall: \( w_n^2(z) := k^2 + m^2 a^2(z) - \frac{\dot{a}(z)}{a(z)} \).

Quantum field fluctuations

Do the amplitudes \( \hat{\phi}(x,t) \) of a quantum field necessarily quantum fluctuations?

- Consider a real-valued function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) and a time \( \tau_0 \).

- Then, we define the state \(| \psi \rangle \) as the joint eigenstate of all operators \( \hat{\phi}(x, \tau_0) \) with eigenvalues \( f(x) \):
  \[
  \hat{\phi}(x, \tau_0) | \psi \rangle = f(x) | \psi \rangle
  \]
Expectation value:
\[ \bar{\Phi}(x, y) = \langle \hat{\Phi}(x, y) \rangle = f(x) \langle \hat{f} \rangle = f(x) \]

Variance:
\[ \Delta \Phi^2(x, y) = \langle \hat{\Phi}(x, y)^2 \rangle - \langle \hat{\Phi}(x, y) \rangle^2 = \langle \hat{f}^2(x) \rangle - f(x)^2 \]
\[ = 0 \quad \text{i.e. no fluctuations.} \]

\[ \Rightarrow \] There are no quantum fluctuations of the quantum field \( \Phi \) if the system is in such a state \( |\Psi\rangle \):

But, can such states \( |\Psi\rangle \) occur in practice?

**No:** The reason is that for these states:
\[ \langle \hat{f} \rangle = \infty \] Exercise: Show this.

**Hint:** For these states, \( \Delta \Phi = 0 \), and so \( \Delta \Phi^2 = \infty \)

But \( \hat{f}^2 \) contains a term \( \frac{\pi^2}{\hbar^2} \)

\[ \Rightarrow \] * Even the state \( |\Psi\rangle \) with \( f(x) = 0 \) for all \( x \) has \( \infty \) energy and is, therefore, not accessible.

**Thus,** all finite energy states do possess quantum fluctuations.

---

Excuse:

What is the analogue of this observation in the case of the harmonic oscillator in quantum mechanics?
How to calculate the amount of quantum field fluctuations?

1. It is not realistic to measure all operators \( \hat{\phi}(x) \) individually.

2. Realistically, we could at most hope to measure an average of the values of \( \hat{\phi} \) over regions \( B \subset \mathbb{R}^3 \) of not too small volume \( V = L \times L \times L \):

\[
\langle \hat{\phi}^2 \rangle_B := \int_{\mathbb{R}^3} \hat{\phi}(x, \gamma) W(x) \, d^3x
\]

with "window function" \( W \)

\[
W(x) = \begin{cases} 
\approx 0 & \text{for all } x \notin B \\
\approx V^{-1} & \text{for all } x \in B
\end{cases}
\]

(we'll also allow \( B \) to be spherical)

Water analog:

A ship of size \( B \) averages the sea over the region \( B \).

Observe: Each ship rocks mostly due to the waves of wavelength of about the scale of \( B \).

Example:

[Graph showing the relationship between rocking amplitude and ship size \( B \)]

In QFT:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size:

[Graph showing the relationship between \( \Delta \phi \) and \( L \)]

- Calculate how this relationship is affected by cosmic expansion:
Quantum field fluctuations in FRW spacetime

1. Choose conformal time \( \gamma \) and comoving coordinates \( x \).

2. Choose a region \( B \) of size \( L \times L \times L \).

Note: In proper coordinates, this is a box of increasing size:

\[
\Omega(\gamma) = L a(\gamma) \times a(\gamma) \times a(\gamma) = L
\]

3. Assume that at \( \gamma_0 \) the system's state \( \Omega(\gamma) \) is in the vacuum state:

\[
\Omega(\gamma_0) = \vert \text{vac}_{\gamma_0} \rangle
\]

4. We choose the mode functions \( v_k(\gamma) \) so that \( \text{vac}_{\gamma_0} = \vert 0 \rangle \) with:

\[
\hat{\phi}(\gamma) = \frac{1}{\sqrt{2}} (v_k(\gamma) a_k + \nu_k a_k^*) \quad \text{and} \quad a_k \vert 0 \rangle = 0
\]

\[\Rightarrow\] The expectation value of the region-averaged field at a time \( \gamma > \gamma_0 \):

\[
\overline{\phi}_B(\gamma) = \langle \Omega(\gamma) \hat{\phi}_B(\gamma) \vert \Omega(\gamma) \rangle
\]

\[= \langle \text{vac}_{\gamma_0} \vert \hat{\phi}_B(\gamma) \vert \text{vac}_{\gamma_0} \rangle
\]

\[= \langle 0 \vert \int_{\Omega(\gamma)} \hat{\phi}(x, \gamma) \hat{W}(x) d^3 x \vert 0 \rangle
\]

\[= \langle 0 \vert \frac{1}{a(\gamma)} \int_{\Omega(\gamma)} \hat{x}(x, \gamma) \hat{W}(x) d^3 x \vert 0 \rangle
\]

\[= \frac{1}{a(\gamma)} \int_{\Omega(\gamma)} \left[ \int \left( v_k(\gamma) a_k + \nu_k a_k^* \right) \hat{W}(x) d^3 x \right] e^{i k \cdot x} (2\pi)^{-3} d^3 k d^3 x
\]

\[= 0
\]

\[\Rightarrow\] The average amplitude of \( \phi_B \) vanishes in the vacuum state.
The quantum fluctuations

While $\tilde{\Phi}_b(\eta)$ vanishes, measurements outcomes for $\tilde{\Phi}_b(\eta)$ are not fully predictable because subject to fluctuations around zero with this standard deviation:

$$\Delta \Phi_b^2(\eta) = \langle 0 | (\tilde{\Phi}_b(\eta) - \bar{\Phi}_b(\eta))^2 | 0 \rangle = \langle 0 | \tilde{\Phi}_b(\eta)^2 | 0 \rangle = \frac{1}{a(\eta)^2} \langle 0 | \left( \int_{\mathbb{R}^3} \tilde{\chi}(x, \eta) W(x) d^3x \right)^2 | 0 \rangle$$

$$= \ldots$$  Exercise: fill in the steps

$$= \frac{1}{2 a(\eta)^2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |V_b(\eta)|^2 \left| \widehat{W}(k) \right|^2 d^3k$$  Fourier transform of the random function $W(x)$.

Assume now for simplicity that $b$ is spherical with radius $L$. Then use spherical coordinates:

$$= \frac{1}{2 a(\eta)^2} \frac{1}{(2\pi)^3} \int_0^\infty k^2 4\pi |V_b(\eta)|^2 \left| \widehat{W}(k) \right|^2 dk$$

Notice the dimension dependence of the integral measure!

Approximation: Consider that:

Typical scale is $L$
(using Fourier) \[ \Rightarrow \text{We can assume that, roughly:} \]

\[ \tilde{W}(k) \approx 0 \text{ for } |k| > \frac{2\pi}{L} \]

\[ \frac{\sin(kc)}{kc} \]

\[ \text{Example: If } W(x) = \frac{1}{L} \quad \text{then } \tilde{W}(k) = \]

and we approximate that \[ \tilde{W}(k) \approx \]

\[ \Rightarrow \Delta \phi_\beta^2(\eta) \approx \frac{1}{4\pi^2a(\gamma)^2} \int_0^{2\pi/L} k^2 \left| V_k(\eta) \right|^2 dk \]

In the integral, the values of \( |V_k(\eta)|^2 \) for small \( k \) are suppressed by \( k^2 \).

\[ \Rightarrow \text{Can approximately replace } |V_k(\eta)| \text{ by its value at } k = \frac{2\pi}{L} : \]

\[ \Delta \phi_\beta^2(\eta) \approx \frac{1}{4\pi^2a(\gamma)^2} \int_0^{2\pi/L} k^2 \left| \frac{V_{2\pi/L}(\eta)}{a(\eta)} \right|^2 dk \]

\[ \Delta \phi_\beta^2(\eta) \approx \frac{1}{4\pi^2} \frac{(2\pi)^3}{3L^3} \left| \frac{V_{2\pi/L}(\eta)}{a(\eta)} \right|^2 \]

\[ = \frac{2\pi}{3L^3} \left| \frac{W_{2\pi/L}(\eta)}{a(\eta)} \right|^2 \]

Here, \( \phi \) is the mode function of \( \phi \), because \( \frac{\phi(\eta)}{a(\eta)} = \phi(\eta) \).
Conclusion:

Assume that at a time \( \eta_0 \) the vacuum state was the state \( \text{loc} \) corresponding to \( \frac{1}{2} V_{\kappa} \).

Then, the typical amplitude of fluctuations of size \( L \) at an arbitrary time \( \eta \) is:

\[
\Delta \phi_{\kappa}^2(\eta) \approx \frac{2\pi}{3L^3} \left| \frac{V_{2\kappa}(\eta)}{a(\eta)} \right|^2 = \frac{2\pi}{3L^3} \left| \frac{W_{2\kappa}(\eta)}{L} \right|^2
\]

Special case: Minkowski space (massless field)
- \( \nu_{\kappa}(\eta) = \nu_{\kappa}(\eta) \)
- \( \eta = t \)
- \( |\nu_{\kappa}(t)|^2 = \frac{1}{\omega_{\kappa}(t)} = \frac{1}{|k|} = \frac{L}{2\pi} \)

\[
\Rightarrow \Delta \phi_{\kappa}^2 \approx \frac{1}{3L^2} \Rightarrow \Delta \phi_{\kappa} \approx \sqrt{\frac{1}{L}}
\]

How to describe field quantum fluctuations using correlators

A primer on classical fluctuations:

- Assume \( m(t) \) is a \( \Delta \)-bandlimited Gaussian white noise signal, i.e., a random signal with Gaussian distributed amplitudes, filtered to leave only frequencies in the interval \((-\Delta, \Delta)\).

- Then, for an ensemble of such noise signals, one can show:

\[
\overline{m(t)} = 0 \quad \forall t
\]

"2-point correlator": \( \overline{m(t)m(t+\Delta t)} = c \frac{\sin(\Omega \Delta t)}{\Omega} \quad \forall t \)

How can we see this?

This noise is ergodic, i.e., we could instead average over all \( t \):
\[ n(t) \hat{n}(t+\epsilon) = \int f(t) f(t+\epsilon) \, dt \]

\( \hat{\text{Auto-covariance}} \)

\[ = \int \int \hat{f}(w) \hat{f}(w') e^{iwt} e^{iwt'} \, dt \, dw \, dw' \]

\[ = \frac{1}{2\pi} \int \hat{f}(w) \hat{f}(-w) e^{iwt} \, dw \]

\[ = \frac{1}{2\pi} \int |\hat{f}(w)|^2 e^{iwt} \, dw \]

\( \Rightarrow \) Auto-covariance and power spectrum are a Fourier pair!

Recall: flatness of spectrum means noise is “white”

\( \square \) For white bandlimited noise: \( |\hat{f}(w)|^2 = \frac{\delta(w)}{2\pi} \)

Exercise: Show that its Fourier transform is indeed \( \sin(\pi L)/\pi L \).

The 2-point correlator in \( \mathbb{R}^3 \):

\[ \langle 0| \hat{\phi}(x, \gamma) \hat{\phi}(x+\Delta x, \gamma) |0\rangle = \frac{1}{a(\gamma)^2} \langle 0| \hat{\chi}(x, \gamma) \hat{\chi}(x+\Delta x, \gamma) |0\rangle \]

Exercise: use mode expansion of \( \hat{\chi} \) and spherical coordinates to derive:

\[ = \frac{1}{a(\gamma)^2} \int_{0}^{2\pi} \frac{d\theta}{4\pi^2} \frac{\sin(kL)}{kL} \left| \psi_0(\chi) \right|^2 \] with \( k = |\mathbf{k}|, \ \Delta = |\mathbf{l}| \).

Notice dimension dependence of the integral’s measure!

Observe: \( \frac{\sin(k\ell)}{k\ell} \approx \begin{cases} 1 \text{ for } k < \ell, \\ 0 \text{ for } k \to \ell \end{cases} \)

Observe: \( k^2 \frac{\sin(k\ell)}{k\ell} \) has largest amplitude around \( k = \frac{3\pi}{L} \).
Estimate:

\[ \langle 0 | \hat{\phi}(x, \gamma) \hat{\phi}(x+\lambda, \gamma) | 0 \rangle \approx a(\gamma)^{2}\frac{2\pi \hbar}{h} \frac{k_{x}^{2}}{4 \pi} \frac{1}{|V_{\lambda}(\gamma)|^{2}} \]

\[ \approx a(\gamma)^{2} \frac{k_{x}^{3}}{12 \pi^{2}} \frac{1}{|V_{\lambda}(\gamma)|^{2}} \frac{1}{k_{x}^{2} + \frac{2\pi}{L}} \]

**Special case:** Minkowski space

Mode function:

\[ |V_{\lambda}|^{2} = \frac{1}{|k|} \]

\[ \langle 0 | \hat{\phi}(x, \gamma) \hat{\phi}(x+\lambda, \gamma) | 0 \rangle \approx \frac{1}{3} \frac{1}{L^{2}} \]

\[ \sim \frac{L}{L} \]

We notice: The variance in a box scales like the correlator!

Both are good measures of the fluctuations.

**Definition:** We define the so-called **Fluctuation Spectrum at time** \( \gamma \) **as a function** \( k \):

\[ \delta \phi_{k}(\gamma) = a(\gamma)^{-1} k^{2} \frac{1}{|V_{\lambda}(\gamma)|} \]

\[ k = \frac{2\pi}{L} \]

**Special case:** Minkowski space with massive field:

- **Scale factor:** \( a(\gamma) = 1 \) for all \( \gamma \)
- **Mode functions:**

\[ V_{k}(\gamma) = \frac{1}{\sqrt{\omega_{k}}} e^{i \gamma \omega_{k}} \text{ with } \omega_{k} = \sqrt{k^{2} + m^{2}} \]
The fluctuation spectrum is: (recall \( k = \frac{2\pi}{L} \))

\[
\delta \phi_k = \frac{k^{3/2}}{(\ln^2 + k^2)^{3/4}} \approx \begin{cases} 
  k & \text{for } k \to \infty \\
  \frac{k^{3/2}}{\sqrt{\ln}} & \text{for } k \to 0 
\end{cases}
\]

and as a function of \( \ln k \) is:

\[
\delta \phi_k \approx \frac{(2\pi)^2 L}{L^2 + \ln^2} L^{-3/2}
\]

Notice: The two different scaling behaviors are not clearly visible in this plot.

Recall "Log-log plots":

\[
x := \ln(k), \quad y := \ln(\delta \phi_k)
\]

Here:

\[
\ln \delta \phi_k = \ln \left( \frac{k^{3/2}}{(\ln^2 + k^2)^{3/4}} \right) \approx \begin{cases} 
  \ln k & \text{for } k \to \infty \\
  \frac{\ln k}{(\ln^2)^{-1/2}} & \text{for } k \to 0 \\
  -\frac{1}{2} \ln(\ln) + \frac{3}{2} \ln k
\end{cases}
\]

Thus:

\[
y \approx \begin{cases} 
  x & \text{for } x \to \infty \\
  -\frac{1}{2} \ln(\ln) + \frac{3}{2} x & \text{for } x \to -\infty
\end{cases}
\]
We notice that, in Minkowski space, large scale (i.e., large $L$, small $k$) fluctuations are strongly suppressed, especially of mass $m \to 0$.

Regarding the Infrared (IR):

* The mass $m$ does not matter at very short distances, i.e., in the ultraviolet.

* But, for large $L$ the mass term does help to suppress the quantum fluctuations. $(\delta \phi \sim \frac{1}{\sqrt{L}}, \sim L^{-1})$

* Generally, in phenomena of QFT, the mass of particles tends to play a role only in the infrared, but not in the ultraviolet, which is why in studies of UV phenomena the mass can often be neglected (e.g., for “perturbative power counting”, a method you encounter in renormalization)
Regarding the Ultraviolet (UV)

* We see that RFT predicts divergently large quantum fluctuations to be found in measurements that resolve smaller and smaller regions.

* For small enough regions B the fluctuations $\Delta \phi_8$ in $\phi_8$ would lead to fluctuations in the Klein Gordon energy momentum tensor that are large enough to cause black holes.

$\Rightarrow$ At this scale, $\approx 10^{-35}$ m, the Planck scale, the notion of distance is assumed to break down.

(Accelerators can probe only distances down to about $10^{-9}$ m)