A taste of quantum fields

Intuition:

* Consider water waves:

* Probe them locally with cork:

* Multiple cork's oscillations are correlated
Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization
4. The harmonic oscillators of fields & their vacuum fluctuations

1. Harmonic oscillators

Classical:

- Hamiltonian: $H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}$

- Equations of motion: $p = -\omega^2 q$, $q = p$

- Lowest energy solution: $q(t) = 0$, $p(t) = 0$
  i.e., $H(t) = 0$ for all $t$

- "Nothing moves, with certainty"
Quantum:

As always when quantizing:

- $H$ and Eqs of motion unchanged.

- But, the canonically conjugate pairs of variables (here, $q$ and $p$) no longer commute:

  \[ \hat{H} = \frac{\hat{p}^2}{2} + \frac{w^2}{2} \hat{q}^2 \]

  \[ \dot{\hat{q}} = \hat{p}, \quad \dot{\hat{p}} = -w^2 \hat{q} \]

  \[ [\hat{q}(t), \hat{p}(t)] = i \hbar 1 \]

\[ \Rightarrow \hat{q}(t), \hat{p}(t), \hat{H} \text{ etc are operator-valued.} \]

Lowest energy solution now?

The lowest energy state, $|\psi_0\rangle$, obeys:

\[ \hat{H} |\psi_0\rangle = E_0 |\psi_0\rangle \]

with $E_0 = \frac{1}{2} \hbar w$

We notice:

Lowest energy is elevated! Why?

(Later for quantum fields: $\Rightarrow$ nonzero vacuum energy)
**Lowest energy state $|q_0\rangle$?**

Consider eigenvasis $|q\rangle$ of $\hat{q}$:

$\hat{q}|q\rangle = q|q\rangle$ for $q \in \mathbb{R}$

$\langle q|q'\rangle = \delta(q-q')$

Then, recall:

$\psi_0(q) = \langle q|\psi_0\rangle = \left(\frac{\omega}{\pi \hbar}\right)^{1/4} e^{-\frac{\omega}{2\hbar} q^2}$

---

**Is oscillator at resting position $q=0$?**

In lowest energy state, $|\psi_0\rangle$, we have:

$\bar{q} = \langle \psi_0|\hat{q}|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_0^*(q) q \psi_0(q) \, dq = 0$

i.e. the position expectation vanishes, as in classical mechanics.

**But, there are quantum fluctuations!**

$\Delta q = \langle \psi_0| (\hat{q} - \bar{q})^2 |\psi_0\rangle^{1/2} = \sqrt{\frac{\hbar}{2m}}$

i.e., actual measurements yield values spread around $q=0$.

$\Rightarrow$ plausible why energy is elevated
Plan:

1. Recall harmonic oscillators ✓
2. Relativistic fields
3. 2nd quantization
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2. Relativistic fields

How to make the Schrödinger equation, say

\[ i \hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \Delta \psi(x,t) \quad (5) \]

relativistically covariant?

\[ \text{Laplacian: } \Delta = \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x_{\alpha}^2} \]

Klein & Gordon:

Recall: \( p_i = -i \hbar \frac{\partial}{\partial x_i} \) and \( E = i \hbar \frac{\partial}{\partial t} \), i.e., the Schrödinger equation can be written in this form:

\[ E \Psi = \frac{p^2}{2m} \Psi \], i.e.:

\[ E = \frac{p^2}{2m} \]

i.e. \( E = \frac{1}{2} m x^2 \)

But special relativity demands:

\[ \frac{E^2}{c^2} - p^2 = m^2 c^2 \]  
(Namely: \( p_x p^x = m^2 c^2 \))

i.e.:

\[ (-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + k^2 \Delta) \Psi = m^2 c^2 \Psi \]
This "Klein-Gordon equation" is usually written as:

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta + m^2 \right) \psi = 0 \quad \text{(units chosen so that } c = 1, h = 1) \]

0r, also \((\Delta + m^2) \psi = 0\) with d’Alembertian \(\Delta = \frac{\partial^2}{\partial x^2} - \Delta\)

**Nonrelativistic limit ok?**

Must show that KG eqn reduces to Schrödinger eqn for small momenta:

**Assume KG Eqn., i.e.,** \(\frac{E^2}{c^2} = m^2 c^2 + \vec{p}^2\)

\[\Rightarrow \quad E = \pm \sqrt{m^2 c^4 + \vec{p}^2 c^2}\]

Choose positive energy solution:

\[E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}\]

**Taylor expansion for small \(\vec{p}^2\):** \((or \ large \ c)\)

\[E = mc^2 + \frac{1}{2} \frac{c^2}{\sqrt{\vec{p}^2 c^2 + m^2 c^4}} \left|_{\vec{p}^2 = 0} \right. + O((\vec{p}^4)^2)\]

\[\Rightarrow \quad E = mc^2 + \frac{\vec{p}^2}{2m} + O((\vec{p}^4)^2)\]
For small momenta the K.G. eqn becomes the Schrödinger eqn:

\[ E \psi = \left( \frac{\hbar^2}{2m} + mc^2 \right) \psi \]

i.e.:

\[ i \hbar \frac{\partial}{\partial t} \psi = \left( -\frac{\hbar^2}{2m} \Delta + mc^2 \right) \psi \]

**Note:** We obtain an extra term:

\[ \hat{\mathcal{H}} = \frac{\hbar^2}{2m} \Delta + mc^2 \]

In QM irrelevant: (use Heisenberg picture)

\[ i \hbar \frac{\partial}{\partial t} \hat{\mathcal{J}} = [\hat{\mathcal{J}}, \hat{\mathcal{H}} + \text{const.}] = [\hat{\mathcal{J}}, \hat{\mathcal{H}}] \]

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**Remarks:**

1a) The negative energy solutions spoil the interpretation of the \( \psi(x,t) \) as a probability amplitude density!

1b) This problem is deep and led to quantum field theory, where this is solved in terms of antiparticles.

2a) There are many ways to generalize the Schrödinger equation to obtain a relativistically covariant equation.
26) E. Wigner (1940s): Complete classification of relativistically covariant wave equations:

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<th>Standard wave eqn</th>
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<td>Klein Gordon eqn.</td>
<td>Higgs, Inflaton, \pi^+, \pi^0</td>
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<td>1/2</td>
<td>Dirac eqn.</td>
<td>e^-, quarks, p^+, \bar{p}</td>
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<td>Maxwell eqns.</td>
<td>Photons, gluons</td>
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Note: The complete classification allows and family high spins and distinguishes massive from massless cases. All covariant wave eqns lead to equivalent QFTs. See e.g. textbook on QFT by S. Weinberg.

Higher spins?
- not observed in truly elementary particles.
- appear to lead to inescapable "divergences" in QFT.

Note:
- "Graviton" should be a spin 2 particle.

Plan:

1. Recall harmonic oscillators
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4. Harmonic oscillators in fields \Rightarrow vacuum fluctuations

3. 2nd quantization

- We will 2nd quantize only the Klein Gordon equation because:
- is easiest
- is only case of cosmological significance that we know of (so far).
**Terminology:** We switch from $\psi$ to $\phi$ and call it a "Field".

**Definition:**

The canonically conjugate field $\Pi(x,t)$ to $\phi(x,t)$ is defined as:

$$\Pi(x,t) = \dot{\phi}(x,t)$$

(Analogous to $p_i = q_i$)

**Klein-Gordon equation can now be written in the form:**

$$\ddot{\Pi}(x,t) - \Delta \phi(x,t) + m^2 \phi(x,t) = 0$$

**Notice:**

The K.G. equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right) \phi = 0$$

$k = 1 = c$

does not couple $\text{Re}(\phi)$ to $\text{Im}(\phi)$:

each separately fulfills the K.G. eqn.

$\Rightarrow$ It suffices to study real-valued $\phi$.

Making $\phi$ complex is then straightforward.
Quantization conditions:

\[
\left[ \hat{\phi}(x, t), \hat{\Pi}(x', t) \right] = i \hbar \delta^3(x - x')
\]

\[
\left[ \hat{\chi}_n(t), \hat{\Pi}_n(t) \right] = i \hbar \delta_{mn}
\]

\[
\left[ \hat{\phi}(x, t), \hat{\phi}(x', t) \right] = 0
\]

\[
\left[ \hat{\Pi}(x, t), \hat{\Pi}(x', t) \right] = 0
\]

We keep the equations of motion:

(El)

\[
\hat{\phi}(x, t) = \hat{\Pi}(x, t)
\]

\[
\dot{\chi}_n(t) = \dot{\Pi}_n(t)
\]

(Ed)

\[
\hat{\Pi}(x, t) = -\left( -\Delta + m^2 \right) \hat{\phi}(x, t)
\]

\[
\dot{\Pi}_n(t) = -K_n \chi_n(t)
\]

**Note:** \( \phi^*(x, t) = \phi(x, t) \) now implies hermiticity: \( \hat{\phi}^+(x, t) = \hat{\phi}(x, t) \)

Is there a Hamiltonian for 2nd quantization? Yes!

\[
\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2} \hat{\Pi}^2(x, t) + \frac{1}{2} \hat{\phi}^2(x, t) \left( m^2 - \Delta \right) \hat{\phi}(x, t) \, d^3x
\]

\[
H = \sum_n \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_n^2} + \omega_n^2 \chi_n^2 \right)
\]

**Proposition:**

With this definition of \( \hat{H} \), the Heisenberg equations:

\[
i \hbar \dot{\hat{\phi}}(x, t) = \left[ \hat{\phi}(x, t), \hat{H} \right]
\]

\[
i \hbar \dot{\hat{\Pi}}(x, t) = \left[ \hat{\Pi}(x, t), \hat{H} \right]
\]

\[
i \hbar \dot{\hat{\chi}}_n(t) = \left[ \hat{\chi}_n(t), \hat{H} \right]
\]

yield the proper eqns of motion: E1, E2.
Indeed, e.g.:

\[ 
\text{ih} \, \hat{\phi}(x,t) = \left[ \phi(x,t), \Pi \right] = \left[ \phi(x,t), \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(x',t) + \text{something} \, \hat{\pi} \, dx' \right] 
\]

\[
= \frac{i}{2} \int \left[ \phi(x,t), \hat{\pi}(x',t) \right] \hat{\pi}(x',t) + \hat{\pi}(x',t) \left[ \phi(x,t), \hat{\pi}(x',t) \right] dx' 
\]

\[
= \frac{i}{2} \int \delta^3(x-x') \, \hat{\pi}(x',t) + \hat{\pi}(x',t) \delta^3(x-x') \, dx' = \hat{\pi}(x,t) \checkmark 
\]

**Exercise:** Prove \((\star)\)