Quantization conditions:

\[
\left[ \hat{\phi}(x,t), \hat{\pi}(x',t) \right] = \imath \hbar \delta^3(x-x')
\]

\[
\left[ \hat{\phi}(x,t), \hat{\phi}(x',t) \right] = 0
\]

\[
\left[ \hat{\pi}(x,t), \hat{\pi}(x',t) \right] = 0
\]

\[
\left[ \hat{\psi}_n(t), \hat{\pi}(x,t) \right] = 0
\]

\[
\left[ \hat{\pi}(x,t), \hat{\psi}_n(t) \right] = 0
\]

We keep the equations of motion:

\[
\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \quad \tag{E1}
\]

\[
\dot{\hat{\psi}}_n(t) = \hat{\rho}_n(t)
\]

\[
\dot{\hat{\pi}}(x,t) = -\left( -\Delta + m^2 \right) \hat{\phi}(x,t) \quad \tag{E2}
\]

\[
\dot{\hat{\rho}}_n(t) = -\kappa_n \hat{\psi}_n(t)
\]

Note: \( \Phi^*(x,t) = \Phi(x,t) \) now implies hermiticity: \( \hat{\Phi}^*(x,t) = \hat{\Phi}(x,t) \)
Proposition:

$E_1, E_2$ follow from the Heisenberg eqns

\begin{align*}
    i \hbar \dot{\hat{\phi}}(x,t) &= \left[ \hat{\phi}(x,t), \hat{H} \right] \\
    i \hbar \dot{\hat{\pi}}(x,t) &= \left[ \hat{\pi}(x,t), \hat{H} \right]
\end{align*}

\text{analogous to:}

\begin{align*}
    i \hbar \dot{\hat{a}}(t) &= \left[ \hat{a}(t), \hat{H} \right] \\
    i \hbar \dot{\hat{p}}(t) &= \left[ \hat{p}(t), \hat{H} \right]
\end{align*}

with the QFT Hamiltonian:

\[ \hat{H} = \int_{\mathbb{R}^3} \frac{1}{2} \hat{\pi}^2(x,t) + \frac{\hbar}{2} \hat{\phi}(x,t) \left( m^2 - \Delta \right) \hat{\phi}(x,t) \, d^3x' \]

\[ \hat{H} = \sum_{a} \left( \frac{\hbar^2}{2} + m^2 \right) \phi_a \]

Plan:

1. Recall harmonic oscillators
2. Relativistic fields
3. 2nd quantization
4. Harmonic oscillators in fields \( \Rightarrow \) vacuum fluctuations

4. Harmonic oscillators in quantum fields

\[ \text{From the above, we need to solve 2 equations:} \]

a) The K.L. eqn: \( \left( \frac{\partial^2}{\partial x^2} - \Delta + m^2 \right) \hat{\phi}(x,t) = 0 \)

b) The commutation rels: \( \left[ \hat{\phi}(x,t), \hat{\phi}(x',t) \right] = i \hbar \delta(x-x') \)
Q: How to solve these eqns?

A: Use similarity to harmonic oscillator problem after overcoming a few technical difficulties:

1st Difficulty: (in reducing the QT problem to harmonic oscillators)

In the K.C. equation,

\[ \hat{\phi}(x,t) = -(-\Delta + m^2)^{\hat{\phi}} \]

Analogy

\[ \hat{\rho}_s(t) = -k_s \hat{q}_s(t) \]

we notice that \((-\Delta + m^2)\), unlike \(k_s\), is not a number!

Q: Can we “transform” \((-\Delta + m^2)\) into a number?

A: Yes: Fourier transform turns derivatives into numbers!

The local field oscillators are coupled.
\[ \Rightarrow \text{Excitations spread.} \]

The oscillators that are local in momentum space are uncoupled.
\[ \Rightarrow \text{Excitations don’t spread in momentum space.} \]
Fourier transform of the spatial variables $\mathbf{x}$:

**Definition:**

$$\hat{\phi}(k, t) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i \mathbf{x} \cdot \mathbf{k}} \hat{\phi}(\mathbf{x}, t) \, d^3x$$

**Traditional notation:** $\hat{\phi}_k(t) := \hat{\phi}(k, t)$

**Traditional terminology:** $\hat{\phi}_k(t)$ is called the field’s $k$-mode.

**Inverse Fourier transform:**

$$\hat{\phi}(\mathbf{x}, t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i \mathbf{x} \cdot \mathbf{k}} \hat{\phi}_k(t) \, d^3k$$

**Proposition:** (Exercise: show this)

a) $\hat{\mathbf{H}} = \int_{\mathbb{R}^3} \frac{1}{2} \hat{\mathbf{\Pi}}_k(t) \hat{\mathbf{\Pi}}_{k'}(t) + \frac{1}{2} \hat{\phi}_k(t) (k^2 + m^2) \hat{\phi}_{k'}(t) \, d^3k$

b) $[\hat{\phi}_k(t), \hat{\mathbf{\Pi}}_{k'}(t)] = i \hbar \delta^3(k + k')$

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$$

$$[\hat{\mathbf{\Pi}}_k(t), \hat{\mathbf{\Pi}}_{k'}(t)] = 0$$

$$[\hat{\phi}_k(t), \hat{\mathbf{\Pi}}_{k'}(t)] = 0$$

$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$$

**c) $\hat{\phi}_k(t) = \hat{\mathbf{\Pi}}_k(t)$**

$$\dot{\hat{\phi}}_k(t) = -\hbar \frac{\delta \phi_\nu(t)}{\delta \phi_\nu(t)}$$

For each mode $k$, we seem to have a harmonic oscillator with $\omega_k = \sqrt{k^2 + m^2}$. 

Analogous to:

$$\mathbf{H} = \sum \frac{1}{2} \omega_n \hat{\mathbf{p}}_n \cdot \hat{\mathbf{p}}_n + \frac{1}{2} \omega_n \hat{\mathbf{q}}_n \cdot \hat{\mathbf{q}}_n$$
Exercise:

Show that \( a \hat{a} + b \hat{a}^2 \) Heisenberg eqn \( \hat{f}(t) = \frac{i}{\hbar} \left[ \hat{f}(t), \hat{A} \right] \) yields c.

\(^* \) arbitrary, \( \hat{f}(0) = \phi_0 \) or \( f(0) = \phi_0 \)

2nd Difficulty: (in reducing the QFT problem to harmonic oscillators)

We notice that the commutation relations

\[
\left[ \hat{\phi}_k(t), \hat{\pi}_{k'}(t) \right] = i \hbar \delta^3(k + k') \quad \text{and} \quad [\hat{q}_n, \hat{p}_{n'}] = i \hbar \delta_{n,n'}
\]

do not match, because the Kronecker \( \delta \) is only either 0 or 1, unlike the Dirac \( \delta \)!

Idea: If we Fourier series instead, should have discrete values of \( k \), thus Kronecker \( \delta \) for CCR!

Strategy:

1. Put system into a large box \([-L/2, L/2]^3\)
2. Assume (for example) periodic boundary conditions.
   (If box large enough it should not matter here what happens at the boundary of the box)
3. Instead of Fourier transform, we can now use Fourier series.

Terminology: Putting a system in a box is called "Infrared regularization.

because "long" wavelengths are removed.
Infrared regularization:

\[ (k_x, k_y, k_z) = \frac{2\pi}{L} (n_x, n_y, n_z) \quad \text{with} \quad n_x, n_y, n_z \in \mathbb{Z} \]

\[ V = L^3 \quad (\text{Volume of box}) \]

Fourier series expansion coefficients:

\[ \hat{\phi}_k(t) = V^{-1/2} \int_V \hat{\phi}(x,t) e^{-i k \cdot x} \, d^3 x \]

The inverse in the Fourier series:

\[ \hat{\phi}(x,t) = V^{-1/2} \sum_k \hat{\phi}_k(t) e^{i k \cdot x} \]

\[ \text{discrute set of vectors} ! \]

The QFT problem in the box:

a) \[ \hat{H} = \sum_k \frac{\hbar^2}{2m} \hat{\pi}_k^2 + \frac{1}{2} \omega_k^2 \hat{\phi}_k^2 = \sum_k \left( \frac{\hbar^2}{2m} \hat{\pi}_k^2 + \frac{1}{2} \omega_k^2 \hat{\phi}_k^2 \right) \]

\[ \omega_k^2 = k^2 + m^2 \]

\[ \text{Kronecker } \delta \]

b) \[ \left[ \hat{\phi}_k(t), \hat{\pi}_{k'}(t) \right] = i \hbar \delta_{k,k'} \quad \left[ \hat{\phi}_k(t), \hat{\phi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_k(t), \hat{\pi}_{k'}(t) \right] = 0 \quad \left[ \hat{\phi}_k(t), \hat{\pi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_k(t), \hat{\phi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_k(t), \hat{\pi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\phi}_k(t), \hat{\pi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_{k'}(t), \hat{\phi}_k(t) \right] = 0 \]

\[ \left[ \hat{\phi}_k(t), \hat{\phi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_k(t), \hat{\phi}_{k'}(t) \right] = 0 \]

\[ \left[ \hat{\pi}_{k'}(t), \hat{\phi}_k(t) \right] = 0 \]

\[ \dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t) \quad \dot{\hat{\pi}}_k(t) = -\omega_k^2 \hat{\phi}_k(t) \]

\[ \dot{\hat{\phi}}_{k'}(t) = \hat{\pi}_{k'}(t) \quad \dot{\hat{\pi}}_{k'}(t) = -\omega_{k'}^2 \hat{\phi}_{k'}(t) \]
3rd Difficulty: (in reducing the QFT problem to harmonic oscillators)

- **Hermiticity:**

  We notice that \( \hat{\phi}^+_k(x,t) = \hat{\phi}^*(x,t) \), \( \hat{\pi}^+_k(x,t) = \hat{\pi}^*(x,t) \) implies

  \[
  \hat{\phi}^+_k(t) = \hat{\phi}^*_{-k}(t), \quad \hat{\pi}^+_k(t) = \hat{\pi}^*_{-k}(t) \quad (H)
  \]

  (Indeed:
  \[
  \hat{\phi}^+_k(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ikx} \hat{\phi}^*(x,t) dx = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ikx} \hat{\phi}(x,t) dx = \hat{\phi}^*_{-k}(t)
  \]

  But eqns (H) do not match:

  \[
  \hat{\phi}^+_k(t) = \hat{\phi}^*_{-k}(t), \quad \hat{\pi}^+_k(t) = \hat{\pi}^*_{-k}(t)
  \]

  **Namely:** Our \( \hat{\phi}_k, \hat{\pi}_k \) are not hermitian!

- **Correspondingly:**

  The analogy between

  \[
  [\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar \delta_{k,-k'} \quad \text{and} \quad [\hat{q}_m, \hat{p}_{m'}] = i\hbar \delta_{m,m'}
  \]

  suffers from \( \delta_{k,-k'} \) instead of \( \delta_{k,k'} \). (we do have \( [\hat{q}_m, \hat{\pi}_{m'}] = i\hbar \delta_{m,m'} \))

- **Mukhanov:**

  Neglects hermiticity issue and treats the field's oscillators just like ordinary quantum oscillators, but with complex, i.e., non-hermitian amplitudes.
Proper treatment:

Define new variables $\hat{q}_k, \hat{p}_k$, which are proper oscillators:

Eqns of motion: $\dot{\hat{q}}_k = \hat{\dot{q}}_k$, $\dot{\hat{p}}_k = -\omega_k^2 \hat{q}_k$

Canon. comm. rels: $[\hat{q}_k, \hat{p}_{k'}] = i \delta_{kk'}$

Hermiticity: $\hat{q}_k = \hat{q}_k^\dagger$, $\hat{p}_k = \hat{p}_k^\dagger$

Then, try ansatz:

$$\hat{\phi}_k = \frac{1}{2} \left( \hat{q}_k + \hat{q}_{-k} \right) + \frac{i}{2 \omega_k} \left( \hat{p}_k - \hat{p}_{-k} \right) \quad (A)$$

Remark: In practice, it'll be more convenient to work with $a_k, a_k^\dagger$:

With $a_k := \sqrt{\omega_k} \hat{q}_k + \frac{i}{\sqrt{\omega_k}} \hat{p}_k$ the ansatz reads: $\hat{\phi}_k = \frac{1}{\sqrt{2 \omega_k}} (a_k + a_k^\dagger)$

Exercise!

Now, show that ansatz (A) succeeds, i.e., that indeed:

Hamiltonian $\hat{H} = \sum_k \frac{1}{2} \left( \hat{\phi}_k^2 + \frac{1}{\omega_k^2} \hat{q}_k^2 \right) \quad (H)$

Eqns of motion: $\dot{\hat{\phi}}_k = \dot{\hat{\phi}}_k$, $\dot{\hat{\phi}}_k = -\omega_k^2 \hat{\phi}_k$

Canon. comm. rels: $[\hat{\phi}_k, \hat{\phi}_{k'}] = i \delta_{kk'}$

Hermiticity cond.: $\hat{\phi}_k = \hat{\phi}_k^\dagger$, $\hat{\phi}_k = \hat{\phi}_k^\dagger$

Finally, via inverse Fourier series show that:

$$\hat{\phi}(x) = \sqrt{\frac{2}{\pi}} \sum_k \left\{ \cos(xk) \hat{q}_k - \frac{i}{\omega_k} \sin(xk) \hat{p}_k \right\} \quad (B)$$

Remark: Ansatz (A) was not unique! The $x$'s and $p$'s could be more mixed! (H) could be different!
Significance of non-uniqueness?

* Ground state of $q,p_x$ oscillators $\rightarrow$ Vacuum
* This need not be lowest energy state of the QFT Hamiltonian
  $\rightarrow$ Problem of vacuum identification on curved space.
  $\rightarrow$ See later.

For now: We solve, using (A), the QFT eqns of the K.C. field.

Namely, we have now solved:

Equations of motion:

$$\begin{cases}
\hat{\phi}(x,t) = \hat{\pi}(x,t) \\
\hat{\pi}(x,t) = -(\Delta + m^2) \hat{\phi}(x,t)
\end{cases}$$

Hermiticity:

$$\hat{\phi}^+(x,t) = \hat{\phi}^*(x,t), \quad \hat{\pi}^+(x,t) = \hat{\pi}^*(x,t)$$

Comm. Comm. Rels:

$$\left[ \hat{\phi}(x,t), \hat{\pi}(x',t) \right] = i\hbar \delta^4(x-x')$$

---

Example: How to calculate quant. fluid of K.C. field?

1. Solve the system of $\infty$ many quantum harmonic oscillator degrees of freedom

$$\hat{q}_k, \hat{p}_k$$

with

$$\hat{H} = \sum_k \frac{1}{2} \hat{p}_k^2 + \frac{\omega_k^2}{2} \hat{q}_k^2, \quad \omega_k = \sqrt{k^2 + \omega_0^2}$$

for all $k = (k_1, k_2, k_3) = \frac{2\pi}{\lambda}(n_1, n_2, n_3)$ where $n_1, n_2, n_3 \in \mathbb{Z}$

2. Choose a state $|\psi\rangle$ of that quantum system.

Example: The oscillators could all lie in their lowest energy state.
3. Given a state $|\psi\rangle$, we can calculate the probability (amplitude density) for finding arbitrary values $q_k(t), p_k(t)$.

Example:

- In vacuum state, we know that the probability distribution of the $q_k$ (and $p_k$ as well) is Gaussian:

\[ \text{prob}(q_k) \sim e^{-\omega_k q_k^2} \]

4. Given $|\psi\rangle$, calculate the probability distribution of the Fourier coefficients:

\[ \hat{\varphi}_k, \hat{\pi}_k \]

Can do because they are simply linear combinations of the harmonic oscillator variables $q_k, p_k$. (Exercise: calculate)

Example:

- For $|\psi_0\rangle$, since $q_k, p_k$ are Gaussian distributed, also the $\hat{\varphi}_k, \hat{\pi}_k$ are Gaussian distributed:

\[ \text{prob}(\varphi_k) \sim e^{-\omega_k \varphi_k^2} \]

(straightforward but tedious to show)
5. Given the prob. distribution of the $\phi$, use Fourier to obtain prob. distribution of $\Phi(x)$!

Example:

- Consider $\Psi_0$.
- Contour lines of a typical $\phi(x,t)$ drawn from the vacuum’s probability distribution for $\Psi$.
- Draw a field $\phi(x)$ from the above calculated probability distribution for fields $\Phi(x)$.
- Actual draw from that distribution.

The fluctuations trace back to the Fourier coefficients and to the $\hat{q}_x$, $\hat{p}_x$ which fluctuate even in lowest energy state.