Particles in QFT

Back in the Heisenberg picture, to solve the QFT is to solve:

1. The hermiticity conditions:
   \[ \hat{\phi}^+(x, t) = \hat{\phi}(x, t), \quad \hat{\pi}^+(x, t) = \hat{\pi}(x, t) \]

2. The canonical commutation relations:
   \[ [\hat{\phi}(x, t), \hat{\pi}(x', t)] = i\delta(x-x') \]

3. The equations of motion:
   \[ \ddot{\hat{\phi}}(x, t) - \Delta \hat{\phi}(x, t) + m^2 \hat{\phi}(x, t) = 0 \]
   \[ \dot{\hat{\pi}}(x, t) = \hat{\phi}(x, t) \]

To simplify:

- Infrared regularization:
  Box size \( L \times L \times L \) with periodic boundary conditions.
  - Project: uses Dirichlet boundary conditions.

- Then Fourier series expansion:
  \[ \hat{\phi}(x, t) = L^{-3/2} \sum_k \hat{\phi}_k(t) e^{ikx} \]
  \[ k = \frac{2\pi}{L}(m_1, m_2, m_3), \quad m_i \in \mathbb{Z} \]

Obtain:

- \[ \hat{\phi}_k(t) = -\left(k^2 + m^2\right) \hat{\phi}_k(t) \]
  and \[ [\hat{\phi}_k, \hat{\phi}_{k'}] = i\delta_{k-k'} \]

- \[ \hat{H} = \sum_k \hat{H}_k \]
  with \[ \hat{H}_k = \frac{1}{2} \hat{\phi}_k^2(t) + \frac{1}{2} \hat{\phi}_k^2(t) \left(k^2 + m^2\right) \]

  i.e.:
  \[ \hat{H} = \sum_k \left( \frac{1}{2} \hat{\phi}_k^2(t) + \frac{1}{2} \hat{\phi}_k^2(t) \left(k^2 + m^2\right) \right) \]
Crucial observations:

* For each wave vector $k = (k_x, k_y, k_z)$ there is an independent harmonic oscillator with frequency $\omega_k = \sqrt{k^2 + m^2}$

and spectrum $\text{spec}(H_k) = \hbar \omega_k \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\}$.

$\Rightarrow$ The excitation levels of $H_k$ differ by the energy

$$E = \omega_k = \sqrt{k^2 + m^2} \quad (\hbar = 1)$$

* This is also the energy of a particle of momentum $k$!

| Hipothesis: Mode excitation = particle creation |

Does this interpretation work?

Water:

$$\phi(x,t)$$

Probe amplitudes, e.g., with a cork.

Quantum field:

$$\hat{\phi}(x,t)$$

Probe amplitudes, e.g., with atoms.

Use as a detector for the field’s particle (e.g., photons for EM field)

One finds:

- Interpretation works but is acceleration- and curvature-dependent.
- Interpretation simple only in Minkowski space for inertial detectors.
Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state \(|4\rangle\) in which we have

3 particles of momentum \(k_a\) and 7 particles of momentum \(k_b\)?

\(|4\rangle = |n_b=3, m_b=7, \text{all other } n_k=0\rangle

= \left| n_{k_a}=3 \right\rangle \otimes \left| n_{k_b}=7 \right\rangle \left( \prod_{k \neq a, b} |m_k=0\rangle \right)

Energy: \(\hat{H} |4\rangle = \begin{cases} \hbar \omega_a \left( \frac{1}{2} + 3 \right) & \text{if } k = k_a \\ \hbar \omega_b \left( \frac{1}{2} + 7 \right) & \text{if } k = k_b \\ \hbar \omega_k \frac{1}{2} & \text{if } k \neq k_a, k_b \end{cases} |4\rangle

\Rightarrow \hat{H} |4\rangle = \left( 3 \omega_a + 7 \omega_b + \sum_{\text{all } k} \frac{1}{2} \omega_k \right) |4\rangle

And one can have linear combinations:

Which is, e.g., the state \(|\xi\rangle\) in which we have

3 particles of momentum \(k_a\) or 7 particles of momentum \(k_b\),

with probability amplitudes \(\alpha\) and \(\beta\), \(\beta^2 = 1 - \alpha^2\)?

\(|\xi\rangle = \alpha |n_{k_a}=3, \text{other } n_k=0\rangle + \beta |n_{k_b}=7, \text{other } n_k=0\rangle

Notice: This is not a state of fixed particle number!

Remark: Some particle species have a number conservation law, e.g., leptons, i.e., \(\bar{e}, \mu^-, \tau^-, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau\),

(when the antiparticles count negatively).

"Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.
Mechanisms for mode excitation/particle creation?

J.e.: What are mechanisms for exciting harmonic oscillators?

2 types of mechanism: (here, \( \dot{q}(t) \) stands for \( \dot{\hat{q}}(t) \))

we'll begin with this effect

A) A "driving force" shakes the oscillator:

\[
\dot{q}(t) = -\omega^2 q(t) + \dot{J}(t)
\]

\[\text{e.g.:} \]

b) A time dependence of \( \omega \) affects the oscillator:

\[
\dot{\hat{q}}(t') = -\omega^2(t') \hat{q}(t)
\]

And, \( \omega^2(t') \) could even go negative!

\[\text{e.g.:} \]

All occur in QFT:

A) Multiple fields enter into the Hamiltonian and into the equations of motion. Thus, fields provide each other with \( J \) terms, e.g.:

\[
H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda \frac{\phi^*(x,t) \psi(x,t)}{\phi^*(x,t) \psi(x,t)} \, d^3 x
\]

Wave interpretation: Nontrivial interaction of waves of different types of fields

Particle interpretation: The collision of particles happens when their mode oscillators drive another.

\( \rightarrow \) Collisions can create and annihilate particles.

Strongest effects?

When oscillator "resonates" with driving force.

\[\text{E.g.:} \quad \text{It takes high energy particles to make high energy particles}\]
B) The presence of gravity can effectively influence the $\omega_n(t)$.

- Wave interpretation: E.g., cosmic expansion stretches the wavelength $\Rightarrow$ expect $\omega = \omega(t)$ decreases. Time, and also:
  - if wavelength $> 
  \text{horizon then } \omega^2(t) < 0$
  \Rightarrow \text{many harmonic mode "oscillators"}
  (then field amplification but no particle interpretation)

- Particle interpretation:
  Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

- Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

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Case A: Particle creation through external driving of mode oscillators.

For example: Production of photons by an antenna:

- We model the electromagnetic field as a
  Klein Gordon field.
  (The fact that EM fields have polarization and have $m \neq 0$ is not important here)

- Consider an arbitrary mode of the
  electromagnetic field:
    \[ \hat{\phi}(t) \]
  should really be quantized too

- We model the electric current as a given classical
  field $j(x,t)$ whose modes are $j_m(t)$. 
In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2} \hat{p}_k(t)^2 \hat{p}_k(t) + \frac{1}{2} \omega_k^2 \hat{q}_k(t)^2 \hat{q}_k(t) + J_k(t) \hat{q}_k(t)$$

⇒ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

⇒ Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$$

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I Preparation:

- Recall that for all observables $\hat{O}$:

$$\hat{O}(t) = \langle \phi | \hat{U}(t) \hat{O}(t) \hat{U}^+ (t) | \phi \rangle$$

$L$ operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t) = \hat{1}, \quad \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

the original Hamiltonian

≈ "Heisenberg Hamiltonian"

- Schrödinger picture? We write, equivalently:

$$\langle \psi | t \rangle = \langle \psi | \hat{U}(t) \rangle \langle \hat{U}(t) | \psi \rangle$$

Exercise: check!
The dynamics is \[ i \frac{d}{dt} | \psi(t) \rangle = \hat{H}_s(t) | \psi(t) \rangle \]

with Schrödinger Hamiltonian: \[ \hat{H}_s(t) = \hat{U}(t) \hat{H}(t) \hat{U}^\dagger(t) \]

Exercise: check

- We will use, equivalently, the Heisenberg picture:

\[
\tilde{f}(t) = \langle \psi_0 | (\hat{U}^\dagger(t) \hat{f}_0 \hat{U}(t)) | \psi_0 \rangle = \langle \psi_0 | \hat{f}(t) | \psi_0 \rangle
\]

with dynamics:

\[ i \frac{d}{dt} \tilde{f}(t) = [\tilde{f}(t), \hat{H}(t)] \]

II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say \( | \psi \rangle \in \mathcal{H} \) (from measurement to measurement).

- The observables, say \( \hat{A}(t), \hat{f}(t), \) etc., are time-dependent operators in Hilbert space.

- Important implication:

The eigenbases and the eigenvalues of observables such as \( \hat{A}(t) \) and any \( \hat{f}(t) \) depend on time!

\[
\hat{f}(t) | f_n(t) \rangle = f_n(t) | f_n(t) \rangle
\]

\[
\hat{A}(t) | E_m(t) \rangle = E_m(t) | E_m(t) \rangle
\]
Example:

* Assume the driven harmonic oscillator starts out at time $t_1$, in $n$'th energy state, say $|\psi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t)\ |E_n(t_1)\rangle = E_n(t)\ |E_n(t_1)\rangle$$

* State vector of the system stays $|\psi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$, the Hamiltonian and its eigenvector and eigenvalues are

$$\hat{H}(t)\ |E_n(t_1)\rangle = E_n(t)\ |E_n(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1), \ |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

\[\Rightarrow\] At time $t_2$, system is still in state $|\psi\rangle$ and still

$$|\psi\rangle = |E_n(t_1)\rangle$$

but $|\psi\rangle$ is generally no longer in (or any other) energy eigenstate!

In particular:

* Assume system starts out at $t_1$, in lowest energy state (i.e., in vacuum): $|\psi\rangle = |E_0(t_1)\rangle$

* Then if $|\psi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

\[\Rightarrow\] At $t_2$, the system's state $|\psi\rangle$ is not the ground state i.e. not the vacuum state, i.e. photons (e.g. photons) exist at time $t_2$. 
III. Strategy for solving quantized driven harmonic oscillator

\[ \Delta \text{Problem: } \quad \Delta \text{CCR: } \quad [\hat{q}(t), \hat{p}(t)] = i\hbar \]

* Hermiticity: \( \hat{q}^+(t) = \hat{q}(t) \), \( \hat{p}^+(t) = \hat{p}(t) \)

* Hamiltonian: \( \hat{H}(t) = \frac{1}{2m} \hat{p}(t)^2 + \frac{1}{2} \omega^2 \hat{q}(t)^2 - J(t) \hat{q}(t) \)

* Heisenberg eqns: \( i \frac{d}{dt} \hat{\Psi}(t) = \left[ \hat{\Psi}(t), \hat{H}(t) \right] \) yield:

\[ \hat{\Psi}(t) = \hat{\Psi}(t) \]

\[ \hat{\Psi}(t) = -i \frac{\omega^2 \hat{q}(t) + J(t)}{\hbar} \]

\[ \Delta \text{Strategy: } \quad \Delta \text{Combine: } \quad \alpha(t) := \hat{q}(t) + i \beta \hat{p}(t) \quad \text{and eqns of motion simplify.} \]

IV. Determine \( \omega \) and \( \beta \):

\[ \Delta \text{Notice first that once we have } \alpha(t) \text{ we immediately obtain } \hat{q}(t), \hat{p}(t): \quad \text{Use of } \hat{q}^+(t) = \alpha^+(t) + \beta \hat{p}(t) \text{ yields:} \]

\[ \hat{q}(t) = \frac{i}{2\alpha} (\alpha^+(t) + \alpha(t)) \]

\[ \hat{p}(t) = \frac{i}{2\beta} (\alpha^+(t) - \alpha(t)) \]

\[ \Delta \text{Use this to express } [\hat{q}, \hat{p}] = i \text{ in terms of new variable } \alpha(t): \]

\[ \Rightarrow [\alpha(t), \alpha^+(t)] = 2 \alpha \beta \]

For simplicity, we choose \( \beta = \frac{1}{2\alpha} \) so that:

\[ [\alpha(t), \alpha^+(t)] = 1 \]
Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{i}{2} d^2 \left( a^*(t) a(t) + \frac{\omega}{2} \right) + \frac{\omega}{2} \frac{1}{4 d^2} \left( a^*(t) + a(t) \right)^2$$

$$- J(t) \frac{1}{2d^2} \left( a^*(t) + a(t) \right)$$

We notice that the terms $\sim a^*(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$\frac{1}{2} d^2 + \frac{\omega}{2} \frac{1}{4 d^2} = 0$$

Thus, we choose: $d = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

Thus, by definition:

$$a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega \left( a^*(t) a(t) + \frac{1}{d^2} \right) - J(t) \frac{1}{\sqrt{2\omega}} \left( a^*(t) + a(t) \right)$$

Solve for $a(t)$:

The Heisenberg equation $i \dot{\hat{q}}(t) = [\hat{q}(t), \hat{H}(t)]$ reads for $a(t)$:

$$i \dot{a}(t) = \omega a(t) + \frac{i}{\sqrt{2\omega}} J(t)$$

Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

$$a(t) = a_{in} e^{-i \omega t} + \frac{1}{\sqrt{2\omega}} \int_0^t J(t') e^{i \omega (t' - t)} dt'$$

Exercise: verify.
VI. Case of force of finite duration

- Assume \( J(t) = 0 \) for all \( t \notin [0, T] \)

- Define
  \[
  J_0 := \frac{e}{i \omega} \int_0^T J(t') e^{i \omega t'} dt'
  \]

- Then:
  \[
  \alpha(t) = \begin{cases} 
  a_m e^{-i \omega t} & \text{for } t < 0 \\
  \text{see above} & \text{for } t \in [0, T] \\
  (a_m + J_0) e^{-i \omega t} & \text{for } t > T 
  \end{cases}
  \]

Next:

Implications in terms of particle (e.g., photon) production?