The driven harmonic oscillator cont’d:

D. Energy eigenstates

Recall \( \hat{H}(t) = \omega (a(t)^\dagger a(t) + \frac{1}{2}) - \frac{i}{\sqrt{2\omega}} (a(t)^\dagger + a(t)) J(t) \):

\[
\begin{cases}
\omega (a_{in}^* a_{in} + \frac{1}{2}) & \text{for } t < 0 \\
\text{something} & \text{for } 0 \leq t \leq T \\
\omega (a_{out}^* a_{out} + \frac{1}{2}) & \text{for } T < t
\end{cases}
\]

Here, \( a_{in} := a(0) \), \( a_{out} := a(T) \) and \( a_{out} = a_{in} + \int_0^T J(t') e^{i\omega t'} dt' \)

with: \( J_0 := \frac{1}{2\omega} \int_0^T J(t') e^{i\omega t'} dt' \)

* For \( t < 0 \), we diagonalized the Hamiltonian

\( \hat{H}(t) = \omega (a_{in}^* a_{in} + \frac{1}{2}) = \hat{H}_{\text{in}} = \text{const.} \)

by using \( [a_{in}, a_{in}^*] = 1 \) to construct its eigenbasis:

\[ \hat{H}_{\text{in}} | n_{in} \rangle = E_n^{(in)} | n_{in} \rangle \]

Namely:

\[ E_n^{(in)} = \omega (n + \frac{1}{2}), \quad n = 0, 1, 2, 3 \ldots \]

\[ | n_{in} \rangle := \frac{1}{\sqrt{n!}} (a_{in}^*)^n | 0_{in} \rangle \]

Note: The set \( \{ | n_{in} \rangle \} \) is a Hilbert basis of the Hilbert space \( \mathcal{H} \).
By $t > T$, the Hamiltonian has become a constant operator:
\[ H(t) = \omega (a_{out}^+ a_{out} + \frac{1}{2}) = H_{0T} = \text{const}. \]

What are its eigenvectors $|n_{out}\rangle$ and eigenvalues $E_{m}$?

**Observation:**

We have:
\[ [a_{out}, a_{out}^+] = 1 \]

$\Rightarrow$ we can construct the eigenspaces of $H_{0T}$ with the same method as the eigenspaces of $H_{0c}$.

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There is a unique vector $|0_{out}\rangle \in \mathcal{H}$ obeying:
\[ a_{out} |0_{out}\rangle = 0 \]

* We define the set of vectors $\{ |n_{out}\rangle \}$:
\[ |n_{out}\rangle := \frac{1}{\sqrt{n!}} (a_{out}^+)^n |0_{out}\rangle \]

* Proposition:
\[ H_{0T} |n_{out}\rangle = E_n |n_{out}\rangle \text{ with } E_n = \omega (n + \frac{1}{2}) = E_n^{(c)} \]

* Proposition:
The set $\{ |n_{out}\rangle \}$ is a ON Hilbert basis of the Hilbert space $\mathcal{H}$. 
How are the two bases related?

* Recall: Both $|\nu_{\text{in}}\rangle$ and $|\nu_{\text{out}}\rangle$ are ON bases of $\mathcal{H}$.

  $\Rightarrow$ Each basis's vector $|\nu_{\text{in}}\rangle$ is a linear combination of the basis vectors $|\nu_{\text{out}}\rangle$ and vice versa.

* Therefore, in particular:

  There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

  $$|\nu_{\text{in}}\rangle = \sum_n \Lambda_n |\nu_{\text{out}}\rangle$$

  $\Rightarrow$ "Bogoliubov Transformation"

* Meaning of the $\Lambda_n$?

  * Recall: The system's state is frozen in state $|\gamma\rangle = |\nu_{\text{in}}\rangle$.

  * Assume we measure at a time $t > T$ the energy, i.e., we measure

    $$\hat{A}(t) = \omega (a_{\text{out}}^+ a_{\text{out}} + \frac{1}{2})$$

  * What is the probability amplitude for finding the energy eigenvalue $E_n$?

    $\Rightarrow$ Clearly:

    $$\text{prob. amp.} (|\nu_{\text{out}}\rangle \rightarrow |\gamma\rangle) = \langle \nu_{\text{out}} | \gamma \rangle$$

    i.e.:

    $$\text{prob.} (|\nu_{\text{out}}\rangle \rightarrow |\gamma\rangle) = |\langle \nu_{\text{out}} | \gamma \rangle|^2$$
Calculate:

\[ \langle n_{\text{out}} | \gamma \rangle = \langle n_{\text{out}} | 0 \text{in} \rangle = \langle \text{out} | \sum_n | \Lambda_n \text{in} \rangle = \Lambda_n \]

\[ \Rightarrow \text{If the oscillator started in its ground state, then at time } t > T \text{ the probability for finding the oscillator in its } n \text{th excited state is given by:} \]

\[ \text{prob.} (|n_{\text{out}}\rangle \text{ at } t > T) = |\Lambda_n |^2 \]

Remark: In QFT, this will be the prob. for finding n particles after charges and currents \( J(x,t) \) excited the vacuum.

**Calculation of \( \Lambda_n \):**

**Proposition:** \( \Lambda_n = e^{-\frac{1}{2} |J_0|^2} \frac{1}{\sqrt{V_n!}} J_0^n |n_{\text{out}}\rangle \)

**Proof:** The claim is that \( |n_{\text{out}}\rangle = \sum_n e^{-\frac{1}{2} |J_0|^2} \frac{1}{\sqrt{V_n!}} J_0^n |n_{\text{out}}\rangle \).

We need to check that indeed: \( a_{\text{in}} |0_{\text{in}}\rangle = 0 \)

Using \( a_{\text{out}} = a_{\text{in}} + J_0 \), we need to check: \( (a_{\text{out}} - J_0) |0_{\text{in}}\rangle = 0 \)

Indeed:

\[ (a_{\text{out}} - J_0) \sum_n e^{-\frac{1}{2} |J_0|^2} \frac{1}{\sqrt{V_n!}} J_0^n |n_{\text{out}}\rangle \]

\[ = e^{-\frac{1}{2} |J_0|^2} (a_{\text{out}} - J_0) \sum_n J_0^n \frac{1}{\sqrt{V_n!}} \frac{1}{\sqrt{V_n!}} (a_{\text{out}})^n |0_{\text{out}}\rangle \]

\[ = e^{-\frac{1}{2} |J_0|^2} (a_{\text{out}} - J_0) \sum_n J_0^n \frac{1}{\sqrt{V_n!}} \frac{1}{\sqrt{V_n!}} (a_{\text{out}})^n |0_{\text{out}}\rangle \]
\[ e^{-\frac{i}{\hbar} \int_0^t (a_{\text{out}}^\dagger - j_0) e^{\int_0^t a_{\text{out}}^\dagger} |\text{out}\rangle = e^{\frac{i}{\hbar} \int_0^t \left( a_{\text{out}}^{\dagger} e^{\int_0^t a_{\text{out}}^\dagger} - j_0 e^{\int_0^t a_{\text{out}}^\dagger}\right)|\text{out}\rangle \]

\text{using } AB = [A,B] + BA

\[ = e^{\frac{i}{\hbar} \int_0^t \left( \left[ a_{\text{out}}^{\dagger}, e^{\int_0^t a_{\text{out}}^\dagger}\right] + e^{\int_0^t a_{\text{out}}^\dagger} a_{\text{out}} - j_0 e^{\int_0^t a_{\text{out}}^\dagger}\right)|\text{out}\rangle \]

\[ = e^{\frac{i}{\hbar} \int_0^t \left( j_0 - j_0 \right) e^{\int_0^t a_{\text{out}}^\dagger} + e^{\int_0^t a_{\text{out}}^\dagger} |\text{out}\rangle = 0 \]  

\text{Note: In the last step, (\text{x}), we used that: } [a_{\text{out}}^{\dagger}, e^{\int_0^t a_{\text{out}}^\dagger}] = j_0 e^{\int_0^t a_{\text{out}}^\dagger}.

\text{Exercise: Show that, more generally, } [a, a^2] = 1 \text{ implies } [a, f(a^2)] = f'(a^2) \text{ by induction.}

\text{Hint: Show that: } [a, a^2] = 1, [a, a^3] = 2a^2, [a, a^4] = 3a^3, \ldots, [a, a^n] = na^{n-1}

\text{Exercise: Verify that } |\omega\rangle = \frac{1}{\sqrt{\hbar}} e^{-\frac{i}{\hbar} \int_0^t \omega} |\omega\rangle \text{ obeys } \langle 0_{\omega} | \omega\rangle = 1.

\text{Apply this strategy to the mode oscillators in QFT:}

\[ \text{Making waves...} \]

\[ \text{Making EM waves...} \]
Recall:
\[ \hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \hat{p}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2) \hat{\phi}(x,t) + J(x,t) \hat{\phi}(x,t) \right) \, d^3x \]

**Example interpretation:**
* \( \hat{\phi}(x,t) \) may be viewed as a slightly simplified version of the quantum electromagnetic field.
* \( J(x,t) \) may be viewed as a simplified version of a given classical electric charge and current density function.

**Example:**
A (Klein-Gordon) charge traveling a path \( \tilde{x}^i(t) \):
Then: \( J(x,t) = q \delta(x - \tilde{x}(t)) \)

In- and out periods

Let us consider the case where
\[ J(x,t) = 0 \quad \text{for all } t \notin [0,T] \]

\[ \Rightarrow \] It suffices to consider the periods \( t < 0 \) and \( t > T \) in both of which \( J(x,t) = 0 \) (and then to relate the bases).

**The free (i.e., undriven) QFT:** \( t < 0 \) or \( t > T \)
\[ \hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \hat{p}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2) \hat{\phi}(x,t) \right) \, d^3x \]
We need to solve:

\[ \dot{\pi}(x, t) - (\Delta - m^2) \dot{\phi}(x, t) = 0 \]

\[ [\phi(x, t), \pi(x', t)] = i \delta^3(x - x') \]

Fourier transformed,

\[ \hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \phi(x, t) e^{-ik \cdot x} \, dx \]

we need to solve:

\[ \hat{\phi}_k''(t) + \left( k^2 + m^2 \right) \hat{\phi}_k(t) = 0 \quad (\text{EoM}) \]

\[ [\hat{\phi}_k(t), \hat{\pi}_k(t)] = i \delta^3(k + k') \quad (\text{CCRs}) \]

Recall: \( \hat{\phi}^+(x, t) = \hat{\phi}(x, t) \) means \( \hat{\phi}_k(t) = \hat{\phi}^+_k(t) \).

Solution strategy due to Fock:

Proceed analogously to the driven oscillator, e.g., during \( t < 0 \):

- Introduce new variables:

  \[ \begin{align*}
  \text{QM:} & \quad a(t) := \sqrt{\frac{\omega}{2}} \phi(t) + i \frac{1}{\sqrt{2\omega}} \pi(t) \\
  \text{QFT:} & \quad a_k(t) := \sqrt{\frac{\omega}{2}} \hat{\phi}_k(t) + i \frac{1}{\sqrt{2\omega}} \hat{\pi}_k(t)
  \end{align*} \]

- Equation of motion and CCRs:

  \[ \begin{align*}
  \text{QM:} & \quad \dot{a}(t) = -i \omega a(t) \quad [a(t), a^+(t)] = 1 \\
  \text{QFT:} & \quad \dot{a}_k(t) = -i \omega_k a_k(t) \quad [a_k(t), a^+_l(t)] = \delta(k - l)
  \end{align*} \]

Remark: Valid only while no force and while \( \omega \) is constant.
Solution, using an initial condition:

**QM:** \( a(t) = e^{-i\omega t}a_{in} \)

\[ [a_{in}, a^*_m] = 1 \]

**QFT:** \( a_n(t) = e^{-i\omega t}a_{in} \)

\[ [a_{in}, a^*_m] = \delta^{(4)}(k - \nu) \]

\[ a_{in} = \frac{1}{\sqrt{V}} \left( \frac{e^{-i\omega_0 t}}{V \omega_0} a_{in} + \frac{e^{i\omega_0 t}}{V \omega_0} a^*_{in} \right) \]

\[ a_{in} = \frac{1}{\sqrt{V}} \left( \frac{e^{-i\omega_0 t}}{V \omega_0} a_{in} + \frac{e^{i\omega_0 t}}{V \omega_0} a^*_{in} \right) \]  

\[ \hat{\phi}(x, t) = \frac{1}{(2\pi)^{3/2}} \left( \frac{1}{V \omega_0} \left( a_{in} e^{-i\omega_0 t + ikx} + a^*_{in} e^{i\omega_0 t + ikx} \right) \right) \]

\[ \left( \text{i.e., } \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^{3/2} V \omega_0} \left( a_{in} e^{-i\omega_0 t + ikx} + a^*_{in} e^{i\omega_0 t - ikx} \right) \right) \]

The Hilbert space of states:

* Analogous to the case of QM, there is a vector, \( |0_{\infty}\rangle \in \mathcal{H} \), which obeys:

\[ a_{in} |0_{\infty}\rangle = 0 \]

\[ (|0_{\infty}\rangle = \bigotimes_{k} |0_{\infty}\rangle) \]

* The Hamiltonian reads (for \( t < 0 \)):

\[ \hat{H} = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{1}{\pi k} \right)^2 + \frac{1}{2} \pi_k \phi^*_k \phi_k d^3k \]

\[ = \int_{\mathbb{R}^3} \omega_k \left( a^*_m a_{in} + \frac{1}{2} \delta(0) \right) d^3k \]

In a box:

\[ \hat{H} = L^{3/2} \sum_k \omega_k \left( a^*_m a_{in} + \frac{1}{2} \right) \]  

(because \( \delta(4, k) \) is now \( \delta_{k, 0} \))

Notice: The divergence \( \sum_k L^{3/2} \omega_k \frac{1}{2} = \infty \) is an "infrared divergence."
After the driving ends, $t > T$:

* One obtains: $a_k(t) = e^{-i\omega_k t} a_{\text{out}, k}$ with $a_{\text{out}, k} = a_{\text{in}, k} + J_k$

$$J_k := \frac{i}{\sqrt{2\omega}} \int_0^T J_k(t') e^{i\omega t'} dt'$$

Here: $J_k(t)$ is the Fourier transform of $J(x,t)$.

* Construct the anti-basis $\xi |n_{\text{out}} \rangle \gamma$ from:

$$a_{\text{out}, k} |n_{\text{out}} \rangle = 0$$

\[\Rightarrow\] (can calculate, e.g., $|\langle n_{\text{out}, k} | 10_{\text{in}} \rangle|^2$ i.e., the probability for $J(x,t)$ to have created $n$ particles of momentum $k$.

Recall:

Making waves...

Making EM waves...

\[\text{Described by } J(x,t)\]
Upgrade: Give the charge $j(x,t)$ its own dynamics:

\[ H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\partial^2 \phi}{\partial x^2}(x,t) - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t) \dot{\phi}(x,t) \right) d^3x \]

and upgrade it to:

\[ H(t) = \frac{1}{2} \int_{\mathbb{R}^3} 1 \otimes \left( \frac{\partial^2 \phi}{\partial x^2}(x,t) - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t) \otimes \dot{\phi}(x,t) \right) d^3x \]

with

\[ j(x,t) = \hat{\mathbf{e}}(t) \delta(x - \mathbf{x}(t)) \]

The Hilbert space: $\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{atom}} \otimes \mathcal{H}_{\text{field}}$

Simplified notation:

\[ H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\partial^2 \phi}{\partial x^2}(x,t) - \phi(x,t)(\Delta - m^2)\phi(x,t) + j(x,t) \dot{\phi}(x,t) \right) d^3x \]
The charged systems can act as emitters and as receivers of waves:

And, quantumly, they can act as emitters and receivers of particles!

Definition: (Unruh, deWitt):
A "particle", such as a photon, is what a "particle detector", such as an atom, can detect, by getting excited.

With acceleration:

An accelerated atom's charge can excite the field.
This, in turn can excite the atom: the Unruh effect

⇒ An accelerated atom may detect particles even if inertial observers only see the vacuum
⇒ Related to gravity via the equivalence principle
⇒ Related to Hawking radiation.