The Unruh effect

(W.B. Unruh, 1976)

An accelerated ice cube will melt, even in vacuum.

The Unruh effect is the observation, by accelerated observers, of particles, even when the field is in the vacuum state in Minkowski space, i.e., even if inertial observers don’t see particles.

Intuition:

A rigged cork can act as sender and detector.

When accelerated, it gets excited - as if detecting.

Similarly:

If simplified to a 2-level system, we say that we have an Unruh Dualit detector system.

An atom or molecule can also both emit and detect particles. This can serve as the definition of particles.

When accelerated, expect particle emission and detection.
We'll consider detectors at rest and in motion:

* A detector at rest has: \( X^\mu(r) = (r, 0, 0, 0) \)

* Case of constant velocity:

\[
X^\mu(r) = (a \, r, \, \vec{b} \, r)
\]

with \( a^2 - b^2 = 1 \). Exercise: verify

* Case of constant acceleration in the x-direction:

\[
\begin{align*}
X^\mu(r) &= d \, \text{sech} \left( \frac{r}{d} \right) \\
X^1(r) &= d \left( 1 + \text{sech}^2 \left( \frac{r}{2d} \right) \right)^{1/2} \\
X^2(r) &= x^2(r) = 0
\end{align*}
\]

Exercise: verify that \( \dot{x} = \text{const} \) (i.e., for small velocities)

* Show that for \( \gamma = 1 \):

\( \dot{x}^2 = (r, a + b \, v^2) \)

The quantum field

\( \Psi \) We assume that, for an inertial observer, the field \( \phi \) is in the Minkowski vacuum. Recall:

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{\frac{i}{\hbar} \hat{p} \cdot \hat{x}} \, d^3k \quad \text{with} \quad \hat{p}_ \mu = \frac{i}{\hbar} \left( \hat{v}_\mu(x) a_\mu + \hat{v}_\mu k^x a_\mu^x \right)
\]

and \( \hat{v}_\mu(x) = \frac{1}{\sqrt{\omega_\nu}} e^{i \omega_\nu x^\nu} \) with \( \omega_\nu = \sqrt{k^2 + m^2} \).

Thus:

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \left( \frac{1}{\sqrt{\omega_\nu}} e^{i \omega_\nu x^\nu + ik^x x^x} a_\nu + \frac{1}{\sqrt{\omega_\nu}} e^{-i \omega_\nu x^\nu - ik^x x^x} a_\nu^x \right) \, d^3k
\]

Note: \( \hat{\phi}(x) \) acts on Hilbert space \( H_{\text{field}} \).

Next: Consider a system (e.g., an atom) that can detect particles of the Klein Gordon field \( \phi \).
The detector system

- Small, localized system with path $X^d(t)$
  - E.g.: * An atom
  - * An oscillator, such as the diatomic molecule $H_2$.

- First quantized.

- Hamiltonian $H_{\text{detector}}$ acts on Hilbert space $\mathcal{H}_{\text{detector}}$.

- Assume $\text{spec}(H_{\text{detector}}) = \{E_0, E_1, E_2, \ldots \}$ is discrete.

$\Rightarrow$ The total quantum system thus consists of two subsystems, with:

$$H_{\text{total}} = H_0 \otimes 1 + 1 \otimes H_0 + H_{\text{interaction}}$$

- On the total Hilbert space:

$$\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{detector}} \otimes \mathcal{H}_{\text{field}}$$

- The interaction Hamiltonian $H_{\text{interaction}}$ consists of operator of both subsystems, usually:

$$H_{\text{interaction}}(t) = \varepsilon(t) \hat{Q}(t) \phi(x^d(t), \vec{x}(t))$$

- Detector efficiency
- An observable
- The field $\hat{\phi}$

(can also be used of the detector's at the current

- as on/off switch, quantum system
- detector location

- Examples: $H_{\text{total}} = \hat{S}_3(t) \otimes \hat{S}_3(x(t))$
- $\alpha$: $H_{\text{total}} = -\frac{e}{mc} \hat{p}_i \otimes \hat{A}(x(t))$

(see Asymptotic: $\theta, A' \approx 0$)
Time evolution

- If we (realistically) assume that the detector efficiency \( \epsilon(\tau) \) is small, we can use perturbation theory.

- In this case, the \( \hbar \)-step picture of time evolution is convenient:

  * Operators evolve according to

    \[
    \hat{\mathcal{H}}_{\text{tot}} = \hat{\mathcal{H}}_{\text{detector}} \otimes 1 + 1 \otimes \hat{\mathcal{H}}_{\text{field}}
    \]

    For example:

    \[
    \hat{A}(\tau) = e^{i \hat{\mathcal{H}}_{\text{tot}} \tau} (\hat{A}_0 \otimes 1) e^{-i \hat{\mathcal{H}}_{\text{tot}} \tau}
    = e^{i \hat{\mathcal{H}}_{\text{detector}} \tau} \hat{A}_0 e^{-i \hat{\mathcal{H}}_{\text{detector}} \tau} \otimes 1
    \]

- States evolve according to \( \hat{\mathcal{H}}_{\text{int}}(\tau) \), i.e., according to the time evolution operator:

  \[
  \hat{U}(\tau) = T \exp \left( i \int_{\tau_i}^{\tau_f} \hat{\mathcal{H}}_{\text{interaction}}(\tau') d\tau' \right)
  \]

  time-ordering symbol

  In perturbation theory, the operators are time-dependent, evolving according to (8)

Perturbative ansatz

- For small detector efficiency \( \epsilon(\tau) \) we can expand:

  \[
  \hat{U}(\tau) = 1 + i \int_{-\infty}^{\tau} \epsilon(\tau') \hat{A}(\tau') \hat{\phi}(x'(\tau'), \bar{x}(\tau)) d\tau' + \mathcal{O}(\epsilon^2)
  \]

  - Note: We can set \( \tau_i = -\infty \) since we can always switch \( \epsilon(\tau) \) on or off.
**Initial conditions**

- We assume that the quantum field $\hat{\phi}$ starts out in a state $|\phi\rangle$ with $|\phi\rangle = $ Minkowski vacuum, $|\phi\rangle = |0\rangle$, or a 1-particle state $|\phi\rangle = |1\rangle$.

- We assume that the detector starts out in its ground state $|E_0\rangle$.

- Thus, the combined system starts out in the state:

$$|\Psi_{in}\rangle = |E_0\rangle \otimes |\phi\rangle$$

- **Time evolution:**

At time $\tau$ the total system is in the state

$$|\Psi(\tau)\rangle = \hat{U}(\tau) |\Psi_{in}\rangle$$

**Particle creation**

- **The problem:**

What is the probability amplitude that, if we measure at time $\tau$ the detector system will be found to have detected something, i.e., to be in an excited state $|E_n\rangle$?

- To this end, calculate:

$$p(\tau) := \langle E_n | \hat{\Theta} | 0 \rangle |\Psi(\tau)\rangle$$

for an arbitrary and state $|\Psi(\tau)\rangle$ of the quantum field $\hat{\phi}$.

- **Note:** We will see that not all states $|\Psi(\tau)\rangle$ yield a nonzero $p(\tau)$. 
Total detection probability:

\[ p(\infty) \approx \langle E_n | \emptyset \Omega | (1 + i \int_{-\infty}^{\infty} \hat{Q}(\tau) \hat{\phi}(x(\tau)) d\tau) | E_0 \rangle \emptyset \Omega \rangle \]

\[ \text{Note:} \quad \langle E_n | E_0 \rangle = 0 \quad \Rightarrow \quad \text{sum over variables} \]

\[ = i \int_{-\infty}^{\infty} e^{i(E_n - E_0) \tau} \langle E_n | \hat{Q}(\tau) | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \Omega \rangle d\tau \]

Recall:
\[ \hat{Q}(\tau) = e^{i\hbar \omega_0 \tau / \hbar} e^{-i\hbar \omega_0 \tau / \hbar} \]

Recall:
\[ \hat{\phi}(x) = \frac{i}{\hbar \omega_k} \int e^{i\omega_k x - \frac{i\omega_k^2 \hbar}{2}} a_k^+ \mathrm{d}k \]

Note: We can now calculate all absorption and emission processes.

Here: Let's focus on particle detection in the vacuum, \( |\Omega\rangle \leq |0\rangle \):

* In \( \hat{\phi}(x) \), only the terms \( a_k^+ \) can contribute, because \( a_k^+ |0\rangle = 0 \)

* Thus, in \( \mathcal{R}_7 \) only the one-particle components contribute:

\[ \mathcal{R}_7 = \Omega \langle \Omega | 1 \Omega \rangle | \Omega \rangle + \int \Omega \langle \Omega | a_k^+ a_k 1 \Omega \rangle | \Omega \rangle \mathrm{d}k + \int \Omega \langle \Omega | a_k^+ a_k^+ a_k a_k 1 \Omega \rangle | \Omega \rangle \mathrm{d}k + \cdots \]

* Thus, let us consider a \( \mathcal{R}_7 = a_k^+ |0\rangle \):
\[ p(\infty) = i \frac{\langle E_n | a_k | E_\infty \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{i(E_n - E_\infty)\tau} i(\omega_k x'(\tau) - \tilde{\chi} x(\tau))}{\sqrt{2\omega_k}} e^{-\frac{i\omega_k x'(\tau)}{\sqrt{2\omega_k}}} \, d\tau \]

\[ \langle 0 | a_k a_k^\dagger | 10 > = \langle 0 | a_k a_k + \delta^3(k-k) | 10 > = \delta^3(k-k) \]

\[ p(\infty) = i \frac{\langle E_n | a_k | E_\infty \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{i(E_n - E_\infty)\tau} i(\omega_k x'(\tau) - \tilde{\chi} x(\tau))}{\sqrt{2\omega_k}} e^{-\frac{i\omega_k x'(\tau)}{\sqrt{2\omega_k}}} \, d\tau \]

Special case: \( 10 > \rightarrow 10 > \) and detector inertial:

* Choose the detector's rest frame: \( x^\mu(\tau) = (\tau, 0, 0, 0) \)

* Thus:
\[ p(\infty) = i \frac{\langle E_n | a_k | E_\infty \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{i(E_n - E_\infty)\tau} i(\omega_k x'(\tau) - \tilde{\chi} x(\tau))}{\sqrt{2\omega_k}} e^{-\frac{i\omega_k x'(\tau)}{\sqrt{2\omega_k}}} \, d\tau \]

Assume \( \epsilon(\tau) = 1 \), i.e., "always on".

\[ = i \frac{\langle E_n | a_k | E_\infty \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{i(E_n - E_\infty)\tau} i\omega_k \tau}{\sqrt{2\omega_k}} e^{-\frac{i\omega_k \tau}{\sqrt{2\omega_k}}} \, d\tau \]

\[ \rightarrow 0 \quad \text{for } \omega_k \tau > 0 \]

\[ = i \frac{\langle E_n | a_k | E_\infty \rangle}{(2\pi)^{3/2}} (2\pi)^{1/2} \delta \left( \sqrt{E_n + \omega_k} - \sqrt{E_0} \right) \frac{1}{\sqrt{2\omega_k}} \]

\[ = 0 \]

* No excitation of the detector, as expected.
Special case: $|d\rangle \equiv 107$ and detector non-inertial:

- The probability amplitude for the detector to become excited will depend on the excitation energy $E_{ex} := E_n - E_0$:

$$p(\omega) = \frac{i\langle E_n|\hat{\phi}|E_0\rangle}{(2\pi)^{3/2}\omega_0^{3/2}}\int_{-\infty}^{\infty} e^{i(E_n - E_0)\tau} e^{i(\omega_n x'(\tau) - \tilde{k}\tilde{x}(\tau))} E(\tau) d\tau$$

- A constant
- Fourier factor (i.e. $\tilde{\tau}$ and $\tilde{E}_\tau$)
- Function that is being Fourier transformed
- (if neglecting the constant)

Clearly: For generic, accelerated detectors the function

$$f(\tau) := e^{i(\omega_n x'(\tau) - \tilde{k}\tilde{x}(\tau))} E(\tau)$$

possesses a Fourier transform

$$F(E_x) = \int_{-\infty}^{\infty} e^{i E_x \tau} f(\tau) d\tau, \quad E_x = E_n - E_0$$

which is generally nonzero for positive $E_x$.

$$\Rightarrow p(\omega) \sim F(E_x) \neq 0 \Rightarrow \text{detector does get excited}.$$ (proportional to “Unruh effect” (European notation)) (while also the field gets excited)

$$\Rightarrow \text{Unruh effect}$$
Example: The constantly accelerated detector.

* One finds that the prob. of excitation is identical to the case in which the detector is in a heat bath of temperature $T$ and where $d$ in the acceleration.


Remark: * Note that both the detector and the quantum field become excited. Is energy conservation violated?

* One can show that the energy stems from the accelerating agent.

By thinking of a regular antenna, if the accelerated e-m wave excites the fields, then would get the field.

* It's the case of a system with charge in time-dependent interaction with the field: An antenna where field & system get excited.

Special case: $|1d\rangle = \lambda |\psi\rangle$:

Recall:

$$|\psi\rangle = \int_0^{\infty} \frac{e^{i(E_n-E_d)\tau}}{\sqrt{2\omega_n}} e^{i(E_n-E_d)\tau} \langle E_n | \hat{a}_d^+ | E_o \rangle \langle O | \hat{\phi}(x(z)) |1d\rangle d\tau$$

Prob. example to do for detector to get excited

Recall:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \left( \frac{1}{\sqrt{\omega_n}} e^{-i\omega_n x^2/2} e^{a_+} + \frac{1}{\sqrt{\omega_n}} e^{i\omega_n x^2/2} e^{-a_+} \right) d^3k$$

$$\Rightarrow \text{For } |1d\rangle = |1_1\rangle = a_1^+ |10\rangle, \text{ we can have: }$$

a.) $|1\rangle = |1_0\rangle \text{ or } |1_1, 1_0\rangle \text{ Would mean detector excites the field further}$

i.e., nd only "detects" a particle

b.) $|1\rangle = |10\rangle \text{: Means detector absorbs a particle.}$
Alternative intuition

A monochromatic wave in an inertial frame is not monochromatic for an accelerated observer.
Thus, the accelerated observer's modes are coupled oscillators; he sees wavelengths change.
Their oscillator's ground state is different.

Calculation strategy:

1. Use accelerated observer's mode decomposition.
2. Relate it to inertial observer's mode decomposition.
3. Choose vacuum for the inertial observer.
4. Calculate particle production for accelerated observer analogous to $|\Psi_{in}\rangle \to |\Psi_{out}\rangle$ transform for driven harmonic oscillator's evolution above.