The 'Tadpole' galaxy

≈ 420 M.y. light yrs away

billions of light yrs away!

Age of the universe?

≈ 13.8 billion yrs

www.math.uwaterloo.ca/~akempf/AMATH875-713.shtml

Spacetime's curvature can be seen directly:

HST: ABELL 2218

How to describe spacetime?
A. Math

Strategy: Start with a more "set" of points (events), \( M \)

Then add structure:

- Define open neighborhoods (i.e., a "topology" on \( M \))
- Define "separability" of points (i.e., Hausdorf condition)
- Define "continuity" (preimage of open sets is open)
- Define "differentiability" (via chart change differentiability)

Later:

- Define tangent & tensor spaces

Curvature = nontriviality of parallel transport

Other descriptions of curvature:

- Curvature = sum of angles in triangle \( \neq \pi \)

- Curvature = nontriviality of Pythagoras law

- Curvature = tidal forces. Math of it: sectional curvature

- Curvature \( ? \) = nontrivial sound of object when vibrating

This field is called Spectral Geometry.

Interesting life connects mathematical languages of quantum theory (spectra etc) and general relativity.

We'll cover it to show recent developments: See arxiv:1312.5247 (PRL)

- Curvature \( ? \) = nontrivial entanglement in vacuum fluctuations

See arxiv:1302.3586 (Found. of Phys)
B) Structure and properties of General Relativity

- Equations of motion for scalars, vectors, spinors and curvature
- Symmetries
  - local and global conservation laws, if any!
- Tetrad formulation, GR as a gauge theory
- Singularities, and their unavoidability

C) Applications to cosmology

- Classification of exact solutions
- Models of cosmological matter
- FRW models, while using the tetrad formalism
e.g. for later use in quantum gravity
- Cosmic inflation
- Black holes
Pseudo-Riemannian Differential Geometry

Differentiable Manifolds

(Riemann 1850s, Poincaré 1890s, Whitney 1930s...)

Def: An n-dimensional topological manifold, $M$, is a Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$.

Here:

Def: A topological space, $M$, is a set, together with a specification of subsets $U_i$, which will be called “open subsets”, which must obey $U_i \cap U_j$ is open, and $\bigcup U_i$ is open.

Def: A topological space $M$ is called Hausdorff, if it is separable, i.e., if $x, y \in M$ and $x \neq y$ then $x, y$ are elements of some disjoint open sets.
**Notice:** Now $M$ has a topology consisting of open sets. And, of course, $\mathbb{R}^m$ also does.

**Recall:** If $A, B$ are top. spaces, then $f: A \to B$ is called continuous if $\forall V \subset B, U := f^{-1}(V) (U \text{ open } \implies V \text{ open})$

$\Rightarrow$ We can now express the idea that $M$ is continuously parametrizable:

**Def:** $M$ is called locally homeomorphic to $\mathbb{R}^m$, if each point, $p$, has a neighborhood $U(p)$, and an invertible continuous map $h: U(p) \to \mathbb{R}^m$

**Def:** A local homeomorphism, $h: U \to \mathbb{R}^m$, $U \in M$ (called “domain”) is called a chart of $M$.

For any point $q \in U$ its image $h(q) \in \mathbb{R}^m$ is a set of $n$ numbers $(x_1, x_2, \ldots, x_n)$ called the coordinates of $q$. 
Def: A chart, \( h \), with domain \( U \), is also called a local coordinate system for \( U \).

Def: A collection of charts \( h_a \) with domains \( U_a \) is called an atlas if \( \bigcup_a U_a = M \).

What, if we want to change coordinates, i.e. if we want to re-label the points of (e.g. a subset of) the manifold?
Consider 2 charts $h_1, h_2$, with intersecting domains $U_1 \cap U_2 \neq \emptyset$.

Then, $h_{12} = h_2 \circ h_1^{-1}$ is a continuous change of coordinates map $h_{12} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Notice: For maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ we know what differentiability means!

Strategy: Let us define the differentiability of an atlas through the differentiability of its chart changes.

Def: An atlas is called $C^r$ differentiable, if all its coordinate changes, $h_{12}$, are $C^r$ diffeomorphisms, i.e., $r$ times continuous differentiable.
Strategy: Enlarge atlas so every point of $M$ is in multiple charts. Then, differentiability of $M$ is definable through atlas differentiability.

Def: Given a $C^r$ differentiable atlas $A$, we can generate a maximal $C^r$ differentiable atlas $D(A)$, by adding all charts whose chart changes with charts in $A$ are differentiable.

Def: $D(A)$ is also called a “Differentiable Structure” of class $C^r$ for $M$.

Def: A differentiable manifold of class $C^r$ is a topological manifold with a maximal atlas of class $C^r$, i.e., with a differentiable structure of class $C^r$.

Theorem (Whitney): Every $C^k$ structure with $k > 1$ is $C^1$ equivalent to a $C^\infty$ structure (i.e., there is always a suitable set of charts).

I.e. any differentiable structure can be smoothed. Any lack of higher differentiability is due to unlucky choice of chart.

Def: Since any $C^1$ manifold is also a $C^\infty$ manifold, we also call differentiable manifolds simply smooth manifolds.