Recall:

- The curvature map, R, is defined through:

$$R: \ \xi_1, \xi_2, \xi_3 \rightarrow R(\xi_1,\xi_2)\xi_3 = \Big(\underbrace{\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}}_{= R(\xi_1,\xi_2)}\Big)\xi_3$$

So, "R" can stand for the tensor, the map and this R!

- 1st Bianchi Identity:

$$\sum_{cyclic} R(\xi,\eta)v = \sum_{cyclic}\Big(T(T(\xi,\eta),v) + (\nabla_\xi T)(\eta,v)\Big)$$

- 2nd Bianchi Identity:

$$\sum_{cyclic}\Big((\nabla_\xi R)(\eta,v) + R(T(\xi,\eta),v)\Big) = 0$$

In a chart?

(Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, dx^i bases)

1st Bianchi: $\displaystyle\sum_{(jkl)} R^i{}_{jkl} = 0$

cyclic sum

2nd Bianchi: $\displaystyle\sum_{(klm)} R^i{}_{jkl;m} = 0$

cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view R as a 2 form, which is (1,1) tensor-valued)

- $R^i{}_{jkl} = -R^i{}_{jlk}$
- $R_{ijkl} = -R_{jikl}$
- $R_{ijkl} = R_{klij}$

$$\langle R(\xi,\eta)v, s\rangle = \langle R(\xi,\eta)s, v\rangle$$

$$\langle R(\xi,\eta)v, s\rangle = -\langle R(v,s)\xi, \eta\rangle$$

Contractions of R:

The Ricci Tensor: $\quad R_{jl} := R^i{}_{jil}$

$\Rightarrow$ clearly: $R_{jl}\, dx^j \otimes dx^l \in T_p(M)_2$

The Curvature Scalar: $\quad R := g^{jl} R_{jl}$

Then, 2nd Bianchi identity implies:

$$\left(R_i{}^k - \frac{1}{2}\delta^k_i R\right)_{;k} = 0$$

$\Rightarrow$ The so-called "Einstein tensor" $G_i{}^k := R_i{}^k - \frac{1}{2}\delta_i^k R$ obeys:

$$G_i{}^k{}_{;k} = 0$$

(this property was crucial guidance for Einstein, as we will see)

Recall strategy:

- Specified g $\Rightarrow$ specified distances in M
  $\Rightarrow$ implicitly specified "shape" of M

  Then, alternatively:

- Specified ∇ $\Rightarrow$ specified parallel transport in M
  $\Rightarrow$ specified "shape" of M, namely:
  ∇ specifies Torsion T and Curvature R.

Now assume a manifold is specified by giving a metric g.

There ought to exist a ∇ which describes the same manifold.

How does g determine Γ?

Idea: The parallel transport of vectors η, v must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path γ and any two vector fields η, v that are $\underbrace{\text{parallel transported}}_{\text{(i.e., autoparallel to }\gamma\text{)}}$ along γ, i.e., for which:

$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = 0, \quad \nabla_{\dot\gamma}\, v(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require: $$\frac{d}{dt}\Big(g(\gamma(t))_{bc}\;\eta^b(\gamma(t))\; v^c(\gamma(t))\Big) = 0$$

$$\text{i.e.: } 0 = \underbrace{\dot\gamma^a\left(g_{bc}\,\eta^b v^c\right)_{;a}}_{=\,\frac{d}{dt}(\ldots)\,=\,\nabla_{\dot\gamma}\langle g,\, \eta\otimes v\rangle} \overset{\text{by }\nabla\text{ obeying Leibniz rule}}{=} \dot\gamma^a\Big(g_{bc;a}\,\eta^b v^c + g_{bc}\underbrace{\eta^b{}_{;a}}_{=\,0 \text{ because } \nabla_{\dot\gamma}\eta = 0}v^c + g_{bc}\,\eta^b \underbrace{v^c{}_{;a}}_{=\,0 \text{ because } \nabla_{\dot\gamma} v = 0}\Big)$$

$\Rightarrow$ $0 = g_{bc;a}\,\dot\gamma^a \eta^b v^c$ for all arbitrary $\dot\gamma, \eta, v$!

$\Rightarrow$ Compatibility of ∇ with g means:

$$\boxed{\nabla_{\xi}\, g = 0 \quad \text{for all } \xi}$$

Is there a ∇ for each choice of g? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold (M, g) there exists a unique ∇ that is torsionless and compatible with g, i.e., which obeys $\nabla g = 0$, the Levi-Civita connection.

More generally: $\forall (M, g)$ and a tensor field T with $T^k_{ij} = -T^k_{ji}$ there is a metric-preserving ∇ whose torsion is T.

In a chart: How to obtain the Levi-Civita ∇ from g?

$\nabla g = 0$ means $g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} - g_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha} = 0$ I

i.e. $g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} - g_{\beta\mu}\Gamma^{\beta}{}_{\alpha\nu} = 0$ II

and $g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^{\beta}{}_{\alpha\mu} - g_{\beta\alpha}\Gamma^{\beta}{}_{\nu\mu} = 0$ III

take: $\frac{1}{2}\left(-\mathrm{I} + \mathrm{II} + \mathrm{III}\right)$

$$\Rightarrow \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^{\beta}{}_{\nu\mu}$$

Thus: $\Gamma^{\beta}{}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$

"Levi-Civita" connection or also called "Riemannian" connection.

Upgrade the math:

- Make use of arbitrary bases e_i, θ^i in (co-) tangent spaces: frames
- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate ∇, g, R, T. (Even the Bianchi identities will look simple.)

Now: Assume again that ∇ and g are still unrelated and $T \neq 0$. (possibly)

"Moving frames":

Def: A "moving frame" is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields e_i which, together, at each point $p \in M$ form a basis of $T_p(M)'$.

Def: We denote the dual basis $\{\theta^i\}_{i=1}^{n}$.

It obeys: $\theta^i(e_j) = \delta^i_j$.

Def: For $n=4$ it may be called $\overbrace{\text{vierbein}}^{\text{german: 4 legs.}}$ or tetrad.
(in arb. dimensions: "vielbein" = many legs)

Notice: Each co-vector $\theta^i(x)$ is a 1-form, and $d\theta^i$ is a 2-form!

Def: Collect them in a "Frame": $\theta^i \otimes e_i$, i.e. a (1,0)-tensor valued 1-form

Remark: If we choose e.g. $\theta^i(x) := dx^i$, then $d\theta^i(x) = 0$.

Remark: A general choice for the $\theta^i(x)$ can always be written in the form:

$$\theta^i(x) = \lambda(x)^i{}_j \, dx^j$$

↑ scalar coefficient functions

Def: We denote the expansion coefficients by functions $C^i{}_{jk}$:

$$\boxed{d\theta^i = \overbrace{-}^{\text{convention}}\frac{1}{2} \underbrace{C^i{}_{jk}}_{\substack{\text{coefficient functions} \\ \text{depend on choice of frame}}} \underbrace{\theta^j \wedge \theta^k}_{\substack{\text{basis for space} \\ \text{of all 2-forms}}} \quad \text{with} \quad \underbrace{C^i{}_{jk} = -C^i{}_{kj}}_{\substack{\text{the sym. part would} \\ \text{drop out}}}}$$

Exercise:
Express the $C^i{}_{jk}$ in terms of the $\lambda^i{}_j$.

Coefficients:

- Torsion: $T^i{}_{k\ell} := \langle \theta^i, T(e_k, e_\ell) \rangle$
- Curvature: $R^i{}_{jk\ell} := \langle \theta^i, R(e_k, e_\ell) e_j \rangle$
- Metric: $g_{ik} := g(e_i, e_k) = \langle e_i, e_k \rangle$
- Christoffel: $\Gamma^i{}_{kj} e_i := \nabla_{e_k} e_j$

Consider arbitrary change of frame: (has nothing to do with a change of chart!)

- assume $\bar{\theta}^i(x) = A^i{}_j(x)\, \theta^j(x)$
- then: $\bar{e}_i(x) = (A^{-1})_i{}^j(x)\, e_j(x)$

(because we chose bases that are dual: $\bar{\theta}^i(\bar{e}_j) = \delta^i{}_j$)

Another step towards more abstract formulation:

Tensor-valued p-forms:

Def: A (r,s)-tensor-valued p-form ϕ is an anti-symmetric p-multilinear mapping at each $q \in M$:

$$\phi : \underbrace{T_q(M)' \times \ldots \times T_q'(M)}_{p \text{ factors}} \longrightarrow T_q(M)^r{}_s$$

Def: The p-forms $\phi^{i_1 \ldots i_r}_{j_1 \ldots j_s} := \phi(\theta^{i_1}, \ldots \theta^{i_r}, e_{j_1}, \ldots, e_{j_s})$ are called the component p-forms relative to the basis $\{e_i\}_{i=1}^n$.

Special cases:

- (r,s) tensors are (r,s) tensor-valued 0-forms.
- p-forms are $(0,0)$ tensor-valued forms.

Torsion 2-form:

- We recall that $T(\xi,\eta) = -T(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields ξ, η:

"torsion 2-form" $\longrightarrow$

$$\underbrace{\Theta^i(\xi,\eta)\, e_i}_{\text{the 2 form } \Theta^i \text{ fed 2 vectors to yield a vector}} := T(\xi,\eta)$$

- Given a frame:

$$\Theta^i = \frac{1}{2} T^i{}_{k\ell}\, \theta^k \wedge \theta^\ell$$

using their antisymmetry

Curvature 2-form:

recall that in canonical basis: $R^i{}_{jk\ell} = -R^i{}_{j\ell k}$

- We recall that also $R(\xi,\eta) = -R(\eta,\xi)$

$\Rightarrow$ can define curvature's (1,1) tensor-valued 2-form:

"curvature 2-form" $\longrightarrow$

$$\underbrace{\overbrace{\Omega^i{}_j(\xi,\eta)}^{\text{numbers}}\, e_i}_{\text{tangent vector}} := \underbrace{R(\xi,\eta)\, e_j}_{\text{tangent vector}}$$

Recall: $R: \xi\, \eta\, e_j \to \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

- Given a frame $\{\theta^i\}_{i=1}^n$:

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jk\ell}\, \theta^k \wedge \theta^\ell$$

The connection as a form?

- Nontrivial because:

  1. Christoffels $\Gamma^i{}_{kj} e_i := \nabla_{e_k} e_j$ are not tensors to start with!

  2. $\Gamma^i{}_{kj}$ is not anti-sym. in any indices, so can't be a 2-form (but can be 1-form):

- Define the connection 1-forms $\omega^i{}_j$: $\boxed{\omega^i{}_j := \Gamma^i{}_{kj}\,\theta^k}$

Thus: $\boxed{\underbrace{\nabla_\xi e_j}_{\text{vector}} = \underbrace{\overbrace{\omega^i{}_j(\xi)}^{\text{scalars}}\, e_i}_{\text{vector}}}$ (because $\nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j$)

- Proposition: cov. deriv. for covectors reads

$$\nabla_\xi \theta^i = -\omega^i{}_j(\xi)\,\theta^j$$

Proof: Leibniz rule and using that ∇ and contractions commute

$$0 = \nabla_\xi \underbrace{\langle \theta^i, e_j\rangle}_{=\delta^i_j} = \langle \nabla_\xi \theta^i, e_j\rangle + \langle \theta^i, \nabla_\xi e_j\rangle$$

$$= \langle \nabla_\xi \theta^i, e_j\rangle + \underbrace{\langle \theta^i, \omega^k{}_j(\xi) e_k\rangle}_{=\omega^i{}_j(\xi) \text{ because } \langle\theta^i, e_k\rangle = \delta^i_k} \qquad (*)$$

$\Rightarrow$ indeed:

$$\boxed{\nabla_\xi \theta^i = -\omega^i{}_\ell(\xi)\,\theta^\ell}$$

(contract with $\langle\ ,e_j\rangle$ to verify that this is Eq. $(*)$)

✓

Connection 1-forms are non-tensorial:

Proposition: Under change of frame $\bar{\theta}^i(x) = A^i{}_j(x)\,\theta^j(x)$ the transformation is:

$$\underbrace{\bar{\omega}^a{}_b}_{\text{1-form}} = \underbrace{\underbrace{A^a{}_i}_{\text{functions}}\underbrace{\omega^i{}_j}_{\text{1form}}\underbrace{A^{-1\,j}{}_b}_{\text{functions}}}_{\text{1-form}} - \underbrace{\underbrace{(dA)^a{}_i}_{\text{1form}}\,\underbrace{\overbrace{(A^{-1})^i{}_b}^{\text{matrix inverse}}}_{\text{functions}}}_{\text{1-form}}$$

function

Proof:

$$-\bar{\omega}(\xi)^a{}_b\,\bar{\theta}^b = \nabla_\xi \bar{\theta}^a = \nabla_\xi\left(A^a_b\theta^b\right) \overset{\text{Leibniz rule}}{=} \Big(\underbrace{dA^a_b(\xi)}_{=\xi(A^a{}_b)}\Big)\,\theta^b + A^a_b\nabla_\xi\theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^b{}_b(\xi)\,A^{-1\,b}{}_c\,\bar{\theta}^c - A^a{}_b\,\omega(\xi)^b{}_c\,A^{-1\,c}{}_d\,\bar{\theta}^d \quad \checkmark$$

true for all $\bar{\theta}$ $\Rightarrow$ proposition above.

The "absolute exterior differential" D:

(It generalizes both ∇ and d)

Proposition: (proof, see e.g. Straumann: check tensorial behavior under frame change)

For every (r,s) tensor-valued p-form ϕ there exists a unique (r,s) tensor-valued $(p+1)$ form $D\phi$ whose components relative to $\{\theta^i\}$ are:

$$(D\phi)^{i_1\dots i_r}_{j_1\dots j_s} = \underbrace{d\,\underbrace{\phi^{i_1\dots i_r}_{j_1\dots j_s}}_{p\text{-form}}}_{p+1\text{ form}} + \overbrace{\underbrace{\omega^{i_1}{}_{\ell}}_{\text{1form}}\wedge\underbrace{\phi^{\ell i_2\dots i_r}_{j_1\dots j_s}}_{p\text{-form}}}^{(p+1)\text{-form}} + \dots - \omega^{\ell}{}_{j_1}\wedge\phi^{i_1\dots i_r}_{\ell\dots j_s} - \dots \qquad (*)$$

Proposition: D is an anti-derivation:

degree of ϕ

$$D(\phi\wedge\psi) = D\phi\wedge\psi + (-1)^p\,\phi\wedge D\psi$$

Special cases:

- An ordinary p-form is $(0,0)$ tensor-valued. In this case, clearly:

$$D = d$$

- An ordinary tensor field is a tensor-valued 0-form. In this case:

$$D = \nabla$$

Exercise: Verify

Hint: Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$, then show (*) implies indeed:

$$\phi^{i_1\dots i_r}{}_{j_1\dots j_s;k} = \phi^{i_1\dots i_r}{}_{j_1\dots j_s,k} + \Gamma^{i_1}{}_{kl}\,\phi^{l i_2\dots i_r}{}_{j_1\dots j_s} + \dots - \Gamma^l{}_{kj_1}\,\phi^{i_1\dots i_r}{}_{l j_2\dots j_s} - \dots$$

How are $\omega, g, \Theta, \Omega$ related now?

Proposition: (Exercise: check)

An affine connection ∇ is metric, if and only if $Dg = 0$, i.e., iff:

$$\boxed{\underbrace{dg_{ik}}_{(0,2)\text{ tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0}$$

Theorem: "The Cartan structure equations"

They express torsion and curvature in terms of the connection

In special case of frame $\theta^i = dx^i$:

$T^i{}_{kj} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$

1.) $\Theta^i = d\theta^i + \omega^i{}_j\wedge\theta^j$ i.e. $\Theta^i = D\theta^i$

$= 0$ for metric connection

Torsion $\Theta = \Theta^i e_i$ is $(1,0)$ tensor-valued 2-form

(The frame, $\theta = \theta^i e_i$ is a $\underbrace{(1,0)\text{ tensor-valued}}_{\text{notice the upper index}}$ $\underbrace{\text{1-form}}_{\text{clear}}$.)

$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls}$

2.) $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j$

Proof of 2.:

$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

$$= \nabla_\xi\left(\omega^i{}_j(\eta)e_i\right) - \nabla_\eta\left(\omega^i{}_j(\xi)e_i\right) - \omega^i{}_j([\xi,\eta])e_i$$

$$\overset{\text{Leibnitz}}{=} \left(\xi(\omega^i{}_j(\eta)) - \eta(\omega^i{}_j(\xi)) - \omega^i{}_j([\xi,\eta])\right)e_i$$

Exercise: Fill in all steps.

$$+\left(\omega^i{}_j(\eta)\,\omega^k{}_i(\xi) - \omega^i{}_j(\xi)\,\omega^k{}_i(\eta)\right)e_k$$

$$= d\omega^i{}_j(\xi,\eta)e_i + \left(\omega^i{}_k \wedge \omega^k{}_j\right)(\xi,\eta)e_i$$

true for all $\xi, \eta, e_i \Rightarrow$ ✓

Use of the Cartan Structure equations?

- Allow proof of simple formulation of the Bianchi identities:

  1st Bianchi: $D\Theta^i = \Omega^i{}_j \wedge \theta^j$

  2nd Bianchi: $D\Omega^i{}_j = 0$

- Thus, for metric connection, i.e. when (i.e. "Riemannian", i.e. "Levi Civita", without torsion.)

  $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma^i_{ij} = \Gamma^i_{ji}$)

  then:

$$\boxed{\begin{aligned} \Omega^i{}_j \wedge \theta^j &= 0 \\ D\Omega^i{}_j &= 0 \end{aligned}}$$

Proposition:

- In the case of metric connection, the Cartan equations yield for <u>arbitrary</u> bases:

$C^l_{ki} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^l{}_{ki} = \frac{1}{2}\left(C^l{}_{ki} - g_{is}\, g^{lj} C^s{}_{kj} - g_{ks}\, g^{lj} C^s{}_{ij}\right) + \frac{1}{2}\, g^{lj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

$$d\theta^i = \overbrace{-}^{\text{convention}}\frac{1}{2}\, C^i{}_{jk}\, \underbrace{\theta^j \wedge \theta^k}_{\text{basis for space of all 2 forms}}$$

coefficient functions depend on choice of frame

- In this case, also:

absent in canonical frame

$$R^i{}_{jab} = \Gamma^i{}_{bj,a} - \Gamma^i{}_{aj,b} + \Gamma^i{}_{al}\Gamma^l{}_{bj} - \Gamma^i{}_{bl}\Gamma^l{}_{aj} - \Gamma^i{}_{lj}\, C^l{}_{ab}$$