Recall:

- **The curvature map**, \( \mathcal{R} \), is defined through:
  \[
  \mathcal{R} : \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3 \to R(\mathfrak{g}_3, \mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{g}_1, \mathfrak{g}_2, -\mathfrak{g}_3, \mathfrak{g}_3)
  \]

- **1st Bianchi Identity**:
  \[
  \sum_{\text{cyclic}} R(\mathfrak{g}_1, \mathfrak{g}_2) = \sum_{\text{cyclic}} (\mathcal{T}(\mathfrak{g}_1, \mathfrak{g}_2), \mathfrak{g}_3)
  \]

- **2nd Bianchi Identity**:
  \[
  \sum_{\text{cyclic}} \left( (\mathcal{T}(\mathfrak{g}_1, \mathfrak{g}_2), \mathfrak{g}_3) + R(\mathcal{T}(\mathfrak{g}_1, \mathfrak{g}_2), \mathfrak{g}_3) \right) = 0
  \]

In a chart? 
(Assuming no torsion, and using \( \frac{\partial}{\partial x_i} \), \( dx_i \) bases)

- **1st Bianchi**:
  \[
  \sum_{\text{cyclic sum}} R'_{ijk} = 0
  \]

- **2nd Bianchi**:
  \[
  \sum_{\text{cyclic sum}} R'_{ijk} = 0
  \]

Other useful properties:

- \( R'_{iue} = - R'_{iue} \)
- \( R'_{ijke} = - R'_{ijke} \)
- \( R'_{ijke} = R_{kije} \)

(Note: This antisymmetry will be useful)

- \( \mathcal{R} \) is a 2-form, which is (\(1,1\)) tensor-valued
Contractions of $R$:

**The Ricci Tensor:** $R_{j\ell} := R_i^i \ v_i^j \ v_i^{\ell}$

⇒ clearly: $R_{j\ell} \ dx^j \ dx^\ell \ e_T = T_\ell (m)$

**The Curvature Scalar:** $R := g^{j\ell} R_{j\ell}$

Then, 2nd Bianchi identity implies:

$$\left( R_i^{\ i} - \frac{1}{4} \delta_i^{\ i} R \right)_{j\ell} = 0$$

⇒ The so-called "Einstein tensor" $G_i^{\ i} := R_i^{\ i} - \frac{1}{2} \delta_i^{\ i} R$ obeys:

$$G_i^{\ i}_{j\ell} = 0$$

(\text{this property was crucial for Einstein, so we will see the guidance for Einstein, etc.)}

Recall strategy:

- Specified $g$ ⇒ specified distances in $M$
  ⇒ implicitly specified "shape" of $M$

  Then, alternatively:

- Specified $\triangledown$ ⇒ specified parallel transport in $M$
  ⇒ specified "shape" of $M$, namely:
    - specifies Torsion $T$ and Curvature $R$.

Now assume a manifold is specified by giving a metric $g$.

Then, there ought to exist a $\triangledown$ which describes the same manifold.

How does $g$ determine $M$?
Idea: The parallel transport of vectors $\gamma, \nu$ must be such that their inner product (i.e., their lengths and relative angles) stays constant.

Consider any path $\gamma^t$ and any two vector fields $\gamma, \nu$ that are parallel transported along $\gamma^t$, i.e., for which:

$$\nabla_{\dot{\gamma}} \gamma (\gamma^t) = 0, \quad \nabla_{\dot{\gamma}} \nu (\gamma^t) = 0 \quad \text{for all } t.$$

Then, require:

$$\frac{d}{dt} \left( \left( \gamma^{(\gamma^t)} \right)_{bc} \gamma^b (\gamma^t) \nu^c (\gamma^t) \right) = 0$$

by obeying/abiding rule

because $\nabla_{\dot{\gamma}} \gamma = 0$

because $\nabla_{\dot{\gamma}} \nu = 0$

i.e.:

$$0 = \frac{\partial}{\partial \gamma^t} \left( g_{bc} \gamma^a \nu^c \right)_{ja} = \frac{\partial}{\partial \gamma^t} \left( g_{bc} \gamma^a \nu^c + g_{bc} \gamma^a \nu^c + g_{bc} \gamma^a \nu^c \right)_{ja}$$

$\Rightarrow$ 0 = $g_{bc} \gamma_a \nu^b \nu^c$ for all arbitrary $\gamma, \nu, \nu$.

$\Rightarrow$ Compatibility of $\nabla^\gamma$ with $g$ means:

$$\nabla^\gamma g = 0 \quad \text{for all } \gamma$$

Is there a $\nabla^\gamma$ for each choice of $g$? Indeed:

Fundamental theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold $(M, g)$ there exists a unique $\nabla^\gamma$ that is torsionless and compatible with $g$, i.e., which obeys $\nabla^\gamma g = 0$, the Levi-Civita connection.

More generally: $\nabla^\gamma (M, g)$ and a tensor field $T$ with $T^k_{\ i} = -T^k_{\ i}$ then is a metric-preserving $\nabla$ whose torsion is $T$. 
In a chart: How to obtain the Levi-Civita $\nabla$ from $g$?

$\nabla g = 0$ means $\nabla \gamma_{\mu \nu} - g_{\mu \rho} \Gamma_{\nu \rho}^{\beta} - g_{\nu \rho} \Gamma_{\mu \rho}^{\beta} = 0 \quad I$

i.e. $\nabla g_{\mu \nu} - g_{\mu \rho} \Gamma_{\nu \rho}^{\beta} - g_{\nu \rho} \Gamma_{\mu \rho}^{\beta} = 0 \quad II$

and $\nabla g_{\nu \rho} - g_{\nu \rho} \Gamma_{\mu \rho}^{\beta} - g_{\mu \rho} \Gamma_{\nu \rho}^{\beta} = 0 \quad III$

Take: $\frac{1}{2} (I + II + III)$

$\Rightarrow \frac{1}{2} (\nabla g_{\mu \nu} + \nabla g_{\nu \rho} - \nabla g_{\nu \mu}) = \frac{1}{2} \ g_{\rho \beta} \ \Gamma_{\nu \mu}^{\beta}$

Thus: $\Gamma_{\nu \mu}^{\beta} = \frac{1}{2} g_{\rho \beta} (\nabla g_{\mu \nu} + \nabla g_{\nu \rho} - \nabla g_{\nu \mu})$

- "Levi-Civita" connection or also called "Riemannian" connection.

Upgrade the math:

- Make use of arbitrary bases $e_i, \theta^i$ in (co-) tangent spaces: frames.

- Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate $\nabla, g, R, \mathcal{F}$. (Even the Bianchi identities will look simple.)

Now: Assume again that $\nabla$ and $g$ are still unrelated and $\mathcal{F} \neq 0$.

(possibly)
"Moving frames:"

**Def:** A "moving frame" is a set, \( \{ e_i \}_{i=1}^n \), of contravariant vector fields \( e_i \) which, together, at each point \( p \) can form a basis of \( T_p M \).

**Def:** We denote the dual basis \( \{ \Theta^i \}_{i=1}^n \).

**Remark:** \( \Theta^i (e_j) = \delta^i_j \).

**Def:** For \( n = 4 \) it may be called vierbein or tetrad.

(from n-dimensional "vierbein"=many legs)

**Remark:** Each co-vector \( \Theta^i (x) \) is a 1-form, and \( d \Theta^i \) is a 2-form!

**Def:** Collect them in "Frame": \( \Theta^i \otimes e_i \), i.e. a (1,0)-tensor valued 1-form.

**Remark:** If we choose e.g. \( \Theta^i (x) = dx^i \), then \( d \Theta^i (x) = 0 \).

**Remark:** A general choice for the \( \Theta^i (x) \) can always be written in the form:

\[
\Theta^i (x) = \lambda^i (x)^j dx^j
\]

**Def:** We denote the expansion coefficients by functions \( C^i_{jk} \):

\[
d \Theta^i = -\frac{1}{2} C^i_{jk} \Theta^j \wedge \Theta^k \quad \text{with} \quad C^i_{jk} = -C^i_{kj}
\]

**Exercise:** Express the \( C^i_{jk} \) in terms of the \( \lambda^i \).
Coefficients:

- Torsion: \( T^i_{\kappa \nu} := \langle \Theta^i, \mathcal{T}(e_\kappa, e_\nu) \rangle \)
- Curvature: \( R^i_{\kappa \lambda \nu} := \langle \Theta^i, \mathcal{R}(e_\kappa, e_\lambda) e_\nu \rangle \)
- Metric: \( g_{i \kappa} := g(e_i, e_\kappa) = \langle e_i, e_\kappa \rangle \)
- Christoffel: \( \Gamma^i_{\kappa \lambda} e_i := \nabla_{e_\kappa} e_\lambda \)

Consider arbitrary change of frame: (has nothing to do with)

- assume \( \tilde{\Theta}^i(x) = A^i_j(x) \Theta^j(x) \)

- then: \( \tilde{e}_i(x) = (A^{-1})^i_j(x) e_j(x) \)

(because we chose bases that are dual: \( \Theta^i(\tilde{e}_j) = \delta^i_j \))

Another step towards more abstract formulation:

Tensor-valued p-forms:

**Def:** A \((r,s)\)-tensor-valued \(p\)-form \(\phi\) is an anti-symmetric \(p\)-multilinear mapping at each \(q \in M\):

\[
\phi : T^* q(M)^r \times \ldots \times T^* q(M) \rightarrow T^* q(M)^s
\]

**Def:** The \(p\)-forms \(\phi_{i_1, \ldots, i_p} := \phi(\Theta^{i_1}, \ldots, \Theta^{i_p}, e_{i_1}, \ldots, e_{i_p})\) are called the component \(p\)-forms relative to the basis \(\{e_i, \Theta^i\}\).

**Special cases:**

- \((r,s)\) tensors are \((r,s)\) tensor-valued 0-forms.
- \(p\)-forms are \((0,0)\) tensor-valued forms.
**Torsion 2-form:**

- We recall that \( J(\xi, \eta) = -J(\eta, \xi) \Rightarrow \) can define the torsion's \((1,0)\) tensor-valued 2-form through its action on 2 vector fields \( \xi, \eta \):

  \[
  \Theta^i(\xi, \eta) e_i := J(\xi, \eta)
  \]

  \( \text{the 2-form } \Theta^i \) \( \text{acts on 2 values to yield a vector} \)

- **Given a frame:** using their antisymmetry

  \[
  \Theta^i = \frac{1}{2} J^i_{\ k\ell} \tilde{\Theta}^k \wedge \tilde{\Theta}^\ell
  \]

**Curvature 2-form:**

- We recall that also \( R(\xi, \eta) = -R(\eta, \xi) \Rightarrow \) can define curvature's \((1,1)\) tensor-valued 2-form:

  \[
  \Omega^i(\xi, \eta) e_i := R(\xi, \eta) e_i
  \]

  \( \text{recall: } R: \xi, \eta e_i \rightarrow \nabla_\xi \nabla_\eta e_i - \nabla_\eta \nabla_\xi e_i - \nabla_{[\xi, \eta]} e_i \)

- **Given a frame:**

  \[
  \Omega^i_{\ j} = \frac{1}{2} R^i_{\ j\kappa \ell} \tilde{\Theta}^\kappa \wedge \tilde{\Theta}^\ell
  \]
The connection as a form?

- Nontrivial because:
  1. Christoffels $\Gamma^i_{kj} e^i \neq \nabla^a e^i$
     are not tensors to start with.
  2. $\Gamma^i_{kj}$ is not anti-sym. in any indices, so can't be a 2-form (but can be 1-form):

- Define the connection 1-forms $\omega^i_j$: $\omega^i_j := \Gamma^i_{kj} \theta^k$

Thus:

$$\nabla^a e^i = \omega^i_j(a) e^j$$ (because $\nabla^a e^i = \theta^k \epsilon_{ik} e^j$)

- Proposition: cov. deriv. for covectors reads

$$\nabla^a \theta^i = -\omega^i_j(a) \theta^j$$

Proof: $0 = \nabla^a \langle \theta^i e^j \rangle = \langle \nabla^a \theta^i e^j \rangle + \langle \theta^i, \nabla^a e^j \rangle$

$$= \langle \nabla^a \theta^i, e^j \rangle + \langle \theta^i, \omega^k_j(a) e^k \rangle$$  \(\text{(x)}\)

$= \omega^i_j(a)$ because $\langle \theta^i, e^k \rangle \epsilon^i_k$

$\Rightarrow$ indeed:

$$\nabla^a \theta^i = -\omega^i_j(a) \theta^j$$  \(\text{to verify that this is Eq. (x)}\)
Connection 1-forms are non-tensorial:

**Proposition:** Under change of frame \( \widetilde{\Theta}^i(x) = \mathcal{A}^i_j(x) \Theta^j(x) \)

the transformation is:

\[
\widetilde{\omega}^a_b = \mathcal{A}^i_j \omega^i_c \mathcal{A}^{-1}^j_b - (d \mathcal{A})^a_i \mathcal{A}^{-1}^i_j \mathcal{A}^{-1}^j_b = \mathcal{A}^a_b \Theta^b
\]

**Proof:**

\[
-\widetilde{\omega}(\xi)^a_b \Theta^b = \nabla_{\xi} \Theta^a = \nabla_{\xi} (\mathcal{A}^b_c \Theta^c) = (d \mathcal{A}_b(\xi)) \Theta^b A^a_c \nabla_{\xi} \Theta^c
\]

\[
= d \mathcal{A}^b_c(\xi) \mathcal{A}^{-1}^c_a \Theta^b - A^b_c \omega^c(\xi) A^a_d \Theta^d = \mathcal{A}^a_b \Theta^b
\]

true for all \( \Theta \) ⇒ proposition above.

---

The "absolute exterior differential" D:

(\text{It generalizes both } \nabla \text{ and } d)

**Proposition:**

(proof, see e.g. Struwe: what tensorial behavior under frame change)

For every \((r,s)\) tensor-valued \(p\)-form \(\phi\) there exists a unique \((r,s)\) tensor-valued \((p+1)\) form \(D\phi\) whose components relative to \(\Theta^j\) are:

\[
(D\phi)^i_{j_1 \ldots j_s} = d\phi^i_{j_1 \ldots j_s} + \omega^i_{j_1} \phi^i_{j_2 \ldots j_s} + \ldots
\]

\[
- \omega^i_{j_1} \phi^i_{j_2 \ldots j_s} - \omega^i_{j_1} \phi^i_{j_2 \ldots j_s} - \ldots
\]
Proposition: \( D \) is an anti-derivation:

\[
D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^{\deg \phi} \phi \wedge D\psi
\]

Special cases:

- An ordinary \( p \)-form is \((0,p)\) tensor-valued. In this case, clearly:
  \( D = \partial \)
- An ordinary tensor field is a tensor-valued \( 0 \)-form. In this case:
  \( D = \nabla \)

Exercise: Verify

Hint: Choose frame \( \Theta^i = dx^i \), use \( \omega^i_j = \Gamma^i_{jk} dx^k \), then show \((*)\) implies indeed:

\[
\delta_{i_1...i_k} = \phi^{(i_1...i_k)} + \Gamma_{(i_1...i_k)(j_1...j_l)} dx^j_1 ... dx^j_l - \Gamma_{(i_1...i_k)(j_1)} dx^j_1 dx^j_2 ..., ...
\]

How are \( \omega, g, \Theta, \Omega \) related now?

Proposition: (Exercise: check)

An affine connection \( \nabla \) is metric, if and only if \( Dg = 0 \), i.e., iff:

\[
\delta g_{ij} - \omega_{ij} - \omega_{ki} = 0
\]

Theorem: "The Cartan structure equations"

In special cases of frame \( \Theta^i = dx^i \):

1.) \[ \Theta^i = d\Theta^i + \omega^i_j \wedge \Theta^j \] i.e. \( \Theta^i = d\Theta^i \)

2.) \[ \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \]
Proof of 2.

\[ \Omega^i_{\ j}(\xi, \eta)e_i = \nabla^i_{\ \xi} \nabla^j_{\ \eta} e_j - \nabla^i_{\ \eta} \nabla^j_{\ \xi} e_j - \nabla^i_{\ \xi, \eta} e_j \]

\[ = \nabla^i_{\ \xi} (\omega^j_{\ j}(\eta)e_j) - \nabla^i_{\ \eta} (\omega^j_{\ j}(\xi)e_j) - \omega^j_{\ j}(\xi, \eta) e_j \]

\[ = \left( \xi(\omega^j_{\ j}(\eta)) - \gamma(\omega^j_{\ j}(\xi)) - \omega^j_{\ j}(\xi, \eta) \right) e_i + \left( \omega^i_{\ j}(\xi) \omega^k_{\ k}(\eta) - \omega^i_{\ j}(\xi) \omega^k_{\ k}(\xi) \right) e_k \]

Exercise: Fill in all steps

\[ = d\omega^i_{\ j}(\xi, \eta)e_i + (\omega^i_{\ j} \omega^k_{\ k})(\xi, \eta) e_i \]

true for all \( \xi, \eta, e_i \Rightarrow \checkmark \)

---

Use of the Cartan Structure equations?

- Allow proof of simple formulation of the Bianchi identities:

1st Bianchi:

\[ \mathcal{D} \Omega^i = \Omega^j_{\ \ j} \wedge \Theta^i \]

2nd Bianchi:

\[ \mathcal{D} \Omega^i_{\ \ j} = 0 \]

Thus, for metric connection, i.e., when

\[ dg_{ik} = \omega_{ik} + \omega_{ki} \text{ and } \Theta^i = 0 \]

(same as \( \nabla g = 0 \text{ and } \Gamma_i = \Gamma_i \))

then:

\[ \Omega^i_{\ j} \wedge \Theta^i = 0 \]

\[ \mathcal{D} \Omega^i_{\ \ j} = 0 \]
Proposition:

1. In the case of metric connection, the Cartan equations yield for arbitrary bases:

\[
\Gamma^L_{ki} = \frac{1}{2} \left( C^L_{ki} - g_{ik} g^{ij} C^L_{ij} - g_{ij} g^{ik} C^L_{ij} \right) \\
+ \frac{1}{2} g^{ij} \left( g_{ij, k} + g_{jk, i} - g_{kj, i} \right)
\]

Recall:

\[
d \theta^i = -\frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k
\]

2. In this case, also:

\[
R^i_{jab} = \Gamma^i_{bja} - \Gamma^i_{ajb} + \Gamma^e_{ab} \Gamma^i_{ej} - \Gamma^e_{eb} \Gamma^i_{aj} - \Gamma^e_{aj} \Gamma^i_{eb}
\]