Recall: The nontrivial shape of a manifold reveals itself in several ways:

1. Violation of angle sum law, \( \angle + \beta + 90^\circ \neq 180^\circ \).
   \( \Rightarrow \) Can encode shape through defiend angles (used in some quantum gravity approaches).

2. Violation of Pythagorean law, \( a^2 + b^2 \neq c^2 \).
   \( \Rightarrow \) Can encode shape through metric distances: \((M, g)\).

   \( \Rightarrow \) Can encode shape through affine connection: \((M, \nabla)\).

Observe: Such local descriptions carry redundant information!

Why? Two (pseudo-)Riemannian manifolds \((M, g), (M, g')\) must be considered equivalent, i.e., they are describing the same space-time, if there exists an isometric, i.e., metric-preserving, isomorphism:

\[ \phi: (M, g) \rightarrow (M, g') \]

Recall: \( \phi \) is called metric-preserving if, under the pull-back map

\[ T\phi^*: T_p(M)_2 \rightarrow T_p(M)_2 \]

the metric stays:

\[ T\phi^*_p(g) = g' \]
Interpretation: \( \xi \) can then be considered to be a mere change of chart:

Observe: Every diffeomorphism \( \xi : M \to M \) of a manifold and its fields (via pullback etc) while staying in one chart, \( \varphi \),

\[
\begin{array}{ccc}
(M, g) & \xrightarrow{\varphi} & \mathbb{R}^n \\
\text{(A)} & \xrightarrow{\xi} & \text{R}
\end{array}
\]

is equivalent to a change of chart from \( \varphi \) to \( \psi \):

\[
\begin{array}{ccc}
(M, g) & \xrightarrow{\psi} & \mathbb{R}^n \\
\text{(A)} & \xrightarrow{\varphi} & \text{R}
\end{array}
\]

where \( \psi = \varphi \circ \xi \).

Vice versa: Every change of chart \( \varphi = \psi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n \)

\[
\begin{array}{ccc}
(M, g) & \xrightarrow{\psi} & \mathbb{R}^n \\
\text{(A)} & \xrightarrow{\varphi} & \text{R}
\end{array}
\]

is equivalent to \( \varphi^{-1} \) followed by a diffeomorphism \( \psi : M \to M \), followed by the old chart map \( \varphi \)

\[
\begin{array}{ccc}
(M, g) & \xrightarrow{\psi} & \mathbb{R}^n \\
\text{(A)} & \xrightarrow{\varphi^{-1}} & \text{R}
\end{array}
\]

with \( \psi \circ \varphi^{-1} = \psi \) i.e. with \( \varphi = \psi \circ \varphi^{-1} \) \( (\psi = \varphi^{-1} \circ \psi) \).

Remark: Thus, \( \log T \) is also rate of change under chart change induced by \( \xi \).
Intuition: $(M,g), (M,g')$ that are related by an isometric diffeomorphism are more or less the same shape.

Definition: A (pseudo-) Riemannian structure, say $\mathcal{E}$, is an equivalence class of (pseudo-) Riemannian manifolds which can be mapped into each other via metric-preserving diffeomorphisms, i.e., via changes of coordinates.

$\Rightarrow$ Space(time) will need to be modelled as a (pseudo-) Riemannian structure, $\mathcal{E}$, i.e., as an equivalence class of pairs $(M,g)$.

Problem: These equiv. classes are hard to handle because absence or existence of $\mathcal{E}$ is hard to check!

$\Rightarrow$ One would like to be able to reliably identify exactly one representative $(M,g)$ per class $\mathcal{E}$.

- This would be called a “fixing of gauge”.
- Why would this be useful?

A key example of when gauge fixing needed: **Quantum gravity**

We discussed detecting and describing shape through:

- Defining angles
- Nontrivial metric distances $(M,g)$
- Nontrivial parallel transport $(M,\Gamma)$
Recall: Quantum theory can be formulated in path integral form.

Applied to gravity:

Expect to have to handle path integrals of the type:

\[ \int e^{iS(\Xi)} D\Xi \]

"all Riemannian structures \( \Xi \)"

But what we initially have is, roughly of the form:

\[ \int e^{iS(g)} \delta(?) Dg \quad \text{or} \quad \int e^{iS(\Xi)} \delta(?) D\Xi \]

"all \( g \)" or "all \( \Xi \)"

Here, \( \delta(?) \) should be such that from each equivalence class of the \( g \)'s or the \( \Xi \)'s only exactly one contributes to the path integral.

Much of Quantum Gravity research is concerned with working out suitable \( \delta(?) \) for \( g \)'s or \( \Xi \)'s or other variables formed from them, such as the frame fields (see "loop quantum gravity").

Q: Can one detect and describe a (pseudo-) Riemannian structure \( \Xi \) directly?

A: Possibly yes, using "Spectral Geometry":

\[ \text{Independent of coordinate systems!} \]

Idea: A manifold's vibration spectrum \( \{ \lambda_n \} \) depends only on \( \Xi \)!

Key question of the field of spectral geometry: (Weyl 1911)

Does the spectrum \( \{ \lambda_n \} \) encode all about the shape, i.e., \( \Xi \)?
Remarks:

- It cannot, if $M$ has infinite volume, because then the spectrum of $\Delta$ will become (almost) completely continuous.
- The spectral geometry of pseudo-Riemannian manifolds is still very little developed.

Theorem:

- Assume $(M, g)$ is a compact Riemannian manifold without boundary, $\partial M = \emptyset$.
- Then, each $\text{spec}(\Delta_p)$ is discrete, with finite degeneracies and without accumulation points.

In practice: We can describe any arbitrarily large part of the universe by a compact Riemannian manifold, $(M, g)$ without boundary, $\partial M = \emptyset$.

This allows us to describe, e.g., 3-dim. space at any fixed time (or also 4-dim. spacetime after so-called Wick rotation).

Types of waves (incl. sounds) on $M$:

- Consider $p$-form fields $\omega(x)$ on $M$, with time evolution, e.g.:
  1. Schrödinger equation: $i \hbar \partial_t \omega(x,t) = -\frac{\hbar^2}{2m} \Delta_p \omega(x,t)$
  2. Heat equation: $\partial_t \omega(x,t) = -\Lambda_p \omega(x,t)$
  3. Klein-Gordon (and acoustic) eqn: $-\partial^2_t \omega(x,t) = \beta \Delta_p \omega(x,t)$
Each of them can be solved via separation of variables:

Assume we find an eigenform $\tilde{\phi}(x)$ of $\Delta$ on $M$:

$$\Delta_p \tilde{\phi}(x) = \lambda \tilde{\phi}(x)$$

They exist: Each $\Delta$ is self-adjoint, w.r.t. the inner product $(\phi, \psi) = \int_M \phi \overline{\psi} dV$.

Then: Schrödinger eqn solved by: $\phi(x,t) = e^{i \frac{\lambda}{2m} t} \tilde{\phi}(x)$

Heat eqn solved by: $\phi(x,t) = e^{-d \lambda t} \tilde{\phi}(x)$

Klein Gordon eqn solved by: $\phi_t(x,t) = e^{\pm i \sqrt{\lambda} t} \tilde{\phi}(x)$

$\Rightarrow$ The spectrum $\text{spec}(\Delta_p)$ is the overtime spectrum of $p$-form type waves on the manifold $M$.

Properties of $\text{spec}(\Delta_p)$:

- **Expectation:**
  The spectra $\text{spec}(\Delta_p)$ for different $p$ carry different information about $M$.

  E.g., scalar and vector seismic waves travel (and reflect) differently.

- **But recall also:**
  a) $[\Delta, \ast] = 0$
  b) $[\Delta, \delta] = 0$
  c) $[\Delta, \delta] = 0$

  This will relate $\text{spec}(\Delta_p)$ to $\text{spec}(\Delta_{m,p}), \text{spec}(\Delta_{p+1})$ and $\text{spec}(\Delta_{p-1})$. 
Use $[\Delta, \tau] = 0$:

Assume: $\omega \in \Lambda_p$ and $\Delta \omega = \lambda \omega$.

Define: $\nu := \tau \omega \in \Lambda_{n-p}$

Then:

$\Delta \nu = \Delta \tau \omega = \tau \Delta \omega = \tau \lambda \omega = \lambda \nu$

$\Rightarrow \text{spec}(\Delta_p) = \text{spec}(\Delta_{n-p})$

Next:

Careful utilization of $[\Delta, \delta] = 0$ and $[\Delta, \delta] = 0$ yields much more information about these spectra!

Notice that $\Delta$ maps exact forms $\omega = d\lambda$ into exact forms:

$\Delta \omega = \Delta d\lambda = d\Delta \lambda$

i.e.:

$\Delta : d\Lambda_r \rightarrow d\Lambda_r$

Analogously, $\Delta$ maps co-exact forms $\omega = \delta \beta$ into co-exact forms:

$\Delta \omega = \delta \Delta \beta = \Delta \delta \beta$

i.e.:

$\Delta : \delta \Lambda_r \rightarrow \delta \Lambda_r$

Also: $\Delta$ can map forms into $0$, namely its eigenspace with eigenvalue $0$, denoted $\Lambda^0_r$. $\Lambda^0_r$ is called the space of “harmonic” $p$-forms:

$\Delta : \Lambda^0_r \rightarrow 0$
Thus: $\Delta$ maps $d\Lambda_r$ and $\delta\Lambda_r$ and $\Lambda^0_r$ into themselves.

Are there any other forms that $\Delta$ could act on? No!

**Proposition ("Hodge decomposition"):**

$$\Lambda^r = d\Lambda_{r-1} \oplus \delta\Lambda_{r+1} \oplus \Lambda^0_r$$

(Recall that $\oplus$ implies that the three spaces are orthogonal!)

**Q:** Why useful?

**A:** It means that every eigenvector of $\Delta_r$ is either in $d\Lambda_{r-1}$, or in $\delta\Lambda_{r+1}$, or in $\Lambda^0_r$ but is never a linear combination of vectors in these spaces.

**Proof:** It is clear that $d\Lambda_{r-1} \subset \Lambda^r$ and $\delta\Lambda_{r+1} \subset \Lambda^r$. We need to show the orthogonalities and completeness:

- Show that $d\Lambda_{r-1} \perp \delta\Lambda_{r+1}$:

  Indeed, assume $\omega = d\nu \in \Lambda^r$ and $d\beta = \delta\beta \in \Lambda^r$.

  Then: $$(\omega, d\beta) = (d\nu, \delta\beta) = \int d^r\nu \int d\beta = 0 \checkmark$$

- Show that if $\omega \in \Lambda^r$ and $\omega \perp d\Lambda_{r-1}$ and $\omega \perp \delta\Lambda_{r+1}$ then: $\omega \in \Lambda^0_r$.

  Indeed, assume $\omega \perp d\Lambda_{r-1}$ and $\omega \perp \delta\Lambda_{r+1}$. Then:

  $\forall \delta: (d\delta, \omega) = 0$ i.e. $- (\delta, d\omega) = 0 \Rightarrow d\omega = 0$ i.e. $\omega$ is "co-exact";

  $\forall \beta: (\delta\beta, \omega) = 0$ i.e. $-(\beta, d\omega) = 0 \Rightarrow d\omega = 0$ i.e. $\omega$ is exact;

  $\Rightarrow \Delta\omega = (d\delta + \delta d)\omega = 0 \Rightarrow \omega \in \Lambda^0_r \checkmark$
Show that if \( \omega \in \Lambda^p \) then \( \omega \perp d\Lambda^p \), and \( \omega \perp \delta \Lambda^{p+1} \).

Assume \( \omega \in \Lambda^p \), i.e., \( \Delta \omega = 0 \), i.e., \((\delta \lrcorner d + d \lrcorner \delta) \omega = 0 \).

\[
\Rightarrow (\omega, (d \delta + \delta d) \omega) = 0
\]

\[
\Rightarrow \delta \omega = \delta (\omega, \delta \omega) + (d \omega, d \omega) = 0 \Rightarrow \delta \omega = 0 \text{ and } d \omega = 0.
\]

(i.e., harmonic forms are closed and co-closed but not exact or co-exact)

Thus, \( B_p := \dim (\Lambda^p) \text{ measures topological non-triviality.} \)

The \( B_p \) are called the "Betti numbers."

\[
\Rightarrow \forall \delta \in \Lambda^p, \quad (\delta, \delta \omega) = 0, \text{ i.e., } (d \delta, \omega) = 0.
\]

\[
\Rightarrow \omega \perp d \Lambda^p, \quad \checkmark
\]

Also: \( \forall \beta \in \Lambda^{p+1}, \quad (\beta, d \omega) = 0, \text{ i.e., } (\delta \beta, \omega) = 0. \)

\[
\Rightarrow \omega \perp \delta \Lambda^{p+1}, \quad \checkmark
\]

**Conclusion so far:**

In the Hodge decomposition, \( \Lambda_{p-1} = d \Lambda_{p-2} \oplus \delta \Lambda_p \oplus \Lambda^p \).

\( \Delta \) maps every term into itself, i.e., \( \Delta \) can be diagonalized in each \( d \Lambda_p, \delta \Lambda_p, \Lambda^p \) separately.

\[
\Rightarrow \Delta \text{ has eigenvalues and } \lambda \text{ on each of these subspaces, for all } \lambda:
\]

\[
\text{spec}(\Delta|_{d \Lambda_p}) , \text{ spec}(\Delta|_{\delta \Lambda_p}) , \text{ spec}(\Lambda^p) = \{0\} \ldots
\]

These spectra are related!
**Proposition:** \( \text{spec} \left( \frac{\Delta}{d \Lambda_r} \right) = \text{spec} \left( \Delta \big|_{\delta \Lambda_{r+1}} \right) \)

and for each eigenvector in one there is one in the other.

This means:

\[ \Lambda_{p-1} = d \Lambda_{p-2} \oplus \delta \Lambda_p \oplus \Lambda_{p-1} \]

\[ \Lambda_p = d \Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda_p \]

\[ \Lambda_{p+1} = d \Lambda_p \oplus \delta \Lambda_{p+2} \oplus \Lambda_{p+1} \]

\[ \vdots \]

**Proof:**

Assume: \( \lambda \in \text{spec} \left( \frac{\Delta}{d \Lambda_r} \right) \) with eigenvector \( \omega \in d \Lambda_r \).

Define: \( v := \delta \omega \in \delta \Lambda_{r+1} \).

Then: \( \Delta v = \Delta \delta \omega = \delta \Delta \omega = \lambda \delta \omega = \lambda v \)

\( \Rightarrow \lambda \in \text{spec} \left( \Delta \big|_{\delta \Lambda_{r+1}} \right) \) and \( v \) is the eigenvector.

Conversely:

Assume: \( \lambda \in \text{spec} \left( \Delta \big|_{\delta \Lambda_{r+1}} \right) \) with eigenvector \( \omega \in \delta \Lambda_{r+1} \).

Define: \( v := d \omega \in d \Lambda_r \).

Then: \( \Delta v = \Delta d \omega = d \Delta \omega = \lambda d \omega = \lambda v \)

\( \Rightarrow \lambda \in \text{spec} \left( \frac{\Delta}{d \Lambda_r} \right) \) and \( v \) is the eigenvector. \( \checkmark \)
Re-use $[\Delta, \star] = 0$:

\[
\begin{array}{c}
\Lambda_{p+1} \\
\Lambda_{n-r-1}
\end{array}
\]

**Proposition:**
\[
\star: d \Lambda_r \rightarrow \delta \Lambda_{n-r}
\]

i.e.: $\star$ exact forms $\rightarrow$ co-exact $n-r$-forms

**Proof:** Assume $\omega = d \delta \in d \Lambda_r$

Define $\nu = \star \omega$

\[
\Rightarrow \nu = \star d \delta = (-1)^{r(n-r)} \star d \star \star \delta
\]

\[
= \delta \delta \in \delta \Lambda_{n-r} \text{ for } d = (-1)^{r(n-r)} \star \delta
\]

**Proposition:**
\[
\delta \Lambda_r \rightarrow d \Lambda_{n-r}
\]

**Proof:** Exercise.

Recall: $\star$ preserves the spectrum of $\Delta$ as we showed already.

**Summary:**

\[
\begin{align*}
\Lambda_{p-1} &= d \Lambda_{p-2} \oplus \delta \Lambda_{p} \oplus \Lambda_{p-1} \\
\Lambda_{p} &= d \Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda_{p} \\
\Lambda_{p+1} &= d \Lambda_{p} \oplus \delta \Lambda_{p+2} \oplus \Lambda_{p+1} \\
&\vdots \\
\Lambda_{n-p} &= d \Lambda_{n-p-1} \oplus \delta \Lambda_{n-p+1} \oplus \Lambda_{n-p}
\end{align*}
\]

Now we also found:

\[
\begin{align*}
\Lambda_{p} &= d \Lambda_{p-1} \oplus \delta \Lambda_{p+1} \oplus \Lambda_{p} \\
&\vdots \\
\Lambda_{n-p} &= d \Lambda_{n-p-1} \oplus \delta \Lambda_{n-p+1} \oplus \Lambda_{n-p}
\end{align*}
\]
Example: \( \dim(M) = 3 \)

\[
\Lambda_0 = \delta \Lambda_1 \oplus \Lambda_0
\]

\[
\Lambda_1 = d \Lambda_0 \oplus \delta \Lambda_2 \oplus \Lambda_1
\]

\[
\Lambda_2 = d \Lambda_1 \oplus \delta \Lambda_3 \oplus \Lambda_2
\]

\[
\Lambda_3 = d \Lambda_2 \oplus \Lambda_3
\]

Same color means same spectrum of \( \Delta \).

Conclusion: There is relatively little independent information in the spectra of p-form waves on \( M \).

E.g., when \( \dim(M) = 3 \), then the spectrum of co-vector waves \( \text{spec}(\Delta_{\Lambda_1}) \) has already all information of all these spectra.

Literature: (neglecting literature on detecting boundary shapes from spectra)

Indeed: The spectra of \( \Delta \) do not contain sufficient information in general to uniquely identify the Riemannian structure from the spectra alone.

Examples: Cases have been found of pairs \( (M, g), (\tilde{M}, \tilde{g}) \) that are isospectral for \( \Delta \) on all \( \Lambda_p \) but that are not diffeomorphically isometric!

Nevertheless: All examples are of limited significance:

- manifolds that are locally if not globally isometric, or
- manifolds that are isospectral only w.r.t. some \( \Delta \) or
- manifolds that are discrete pairs (e.g. mirror images).
Fresh approach to spectral geometry (AK)

Strategy: Iterate infinitesimal inverse spectral geometry

Assume both, the manifold and its spectra are given:

\[(M, g)\]

A compact Riemannian manifold \((M, g)\) without boundary

The spectra \(\{\lambda^{(i)}_m\}\) of\n
\(\Delta^{(i)}\) on the manifold.

Could be Laplacians not only on forms but also on general tensors

Perturbation:

Now change the shape of \((M, g)\) slightly, through:

\[g \rightarrow g + h\]

This will slightly change the spectra to:

\[\{\lambda^{(i)}_m\} \rightarrow \{\lambda^{(i)}_m + \mu^{(i)}_m\}\]

Why is this linearization useful?

- One can define a self-adjoint Laplacian \(\Delta^{(m)}\) on \(T^*_2(M)\), with Hilbert basis \(\{b_m(x)\}\) and eigenvalues \(\{\lambda^{(m)}_n\}\):

\[\Delta^{(m)} b_m(x) = \lambda^{(m)}_n b_m(x)\]
The metric's perturbation $h\in T^1_0(M)$ can be expanded:

$$h = \sum_{m=1}^{\infty} h_m b_m(x)$$

The perturbation of $\text{spec}(\Delta^{(m)})$ is:

$$\{\lambda^{(m)}_n\} \rightarrow \{\lambda^{(m)}_n + \mu^{(m)}_n\}$$

We obtain a linear map $S$:

$$S : \{\mu^{(m)}_n\} \rightarrow \{\mu^{(m)}_n\}$$

$$S' : h_m \rightarrow \mu_n = S_{nm} h_m$$

**Notice:** Consider only eigenvectors and eigenvalues up to a cutoff scale. Then, there are as many parameters $\{\mu^{(m)}_n\}$ as $\{\lambda^{(m)}_n\}$.

$S$ is a square matrix.

If $\det(S') \neq 0$, then $S^{-1}$ exists.

Should we be able to iterate the perturbations?

This is ongoing research.
Remarks: □ Not all $h$ actually change the shape:

- If $h = L_\xi g$, for some vector field $\xi$, then $g \to g + h$ is merely the infinitesimal change of chart belonging to the flow induced by $\xi$.

- Symmetric covariant 2-tensors such as $h$ have a canonical decomposition similar to the Hodge decomposition. Thus, $\Delta$ has three spectra on $T^2(M)$.

Reference: See also e.g. the video of my recent talk at PI: [http://pirsa.org/15090062](http://pirsa.org/15090062)

Infinitesimal spectral geometry arose from my paper on how Spacetime could be simultaneously continuous and discrete, in the same way that information can.