Plan:  
I. The dynamics of matter and radiation in curved spacetime
II. Energy-momentum tensor
III. The dynamics of spacetime itself

1. Recall: On a (pseudo-)Riemannian manifold, equations are well-defined only if defined independently of any chart.

⇒ Any eqn, including the eqns of motion for matter fields, must be eqns among tensors and their covariant derivatives.

⇒ Need a tensor field, $\Psi$, for each species of particle: $e^-$, $q$, gluon, $\pi^\pm$, photon, $W^\pm$, etc.

Notation: $\Psi^{a \cdots b}_{(i) c \cdots d}$

Species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p.59)

Question: Could we have also an additional connection field $\Gamma^\mu_{\nu\lambda}$?
Yes, we could: But, the difference field $Q^k_{ij} := \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}$ is actually a tensor field!

\[
\begin{align*}
\Gamma^k_{ab} &\rightarrow \frac{\partial x^k}{\partial x^a} \frac{\partial x^i}{\partial x^b} + \frac{\partial x^k}{\partial x^b} \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \\
\tilde{\Gamma}^k_{ab} &\rightarrow \frac{\partial x^k}{\partial x^a} \frac{\partial x^i}{\partial x^b} + \frac{\partial x^k}{\partial x^b} \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b}
\end{align*}
\]

\Rightarrow \quad (\Gamma^k_{ab} - \tilde{\Gamma}^k_{ab}) \rightarrow \frac{\partial x^k}{\partial x^a} \frac{\partial x^i}{\partial x^b} \frac{\partial x^j}{\partial x^b} (\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})

\Rightarrow \quad Q^k_{ab} \rightarrow \frac{\partial x^k}{\partial x^a} \frac{\partial x^i}{\partial x^b} \frac{\partial x^j}{\partial x^b} Q^k_{ab}

\Rightarrow \text{Introducing an additional connection } \tilde{\Gamma} \text{ is same as introducing simply a new tensor field } Q.

\textbf{Remark:} \Rightarrow \text{"variations" } \delta \Gamma^k_{ab} \text{ will behave tensorially!}

\underline{Eqs of motion of matter fields?}

\textbf{Action principle:} (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, $L$, namely a scalar function of the matter fields $\Psi_{(i)}^{a...b}$ and their first covariant derivatives, and now also of the metric $g$:

\[
L(\Psi) = L^{(\text{matter})}\left(\{\Psi_{(i)}^{a...b}\}, \{\Psi_{(i)}^{a...b}\}, \text{g}\right)
\]
Define the action functional:

\[ S[\Psi] := \int_B \left( \sum_{\text{scalar}} L(\Psi) + \sum_{\text{volume form}} \text{some bounded and closed } n\text{-dim region in } M \right) \ d^n x \in \mathbb{R} \]

Thus, each physical field \( \Psi(x) \) (as a function of both space and time) is mapped into a number \( S[\Psi] \).

**Action principle (or postulate) of classical physics:**

In nature, physical fields \( \Psi \) are such that \( S[\Psi] \) is extremal in the space of all fields \( \Psi \).

Thus: The matter fields \( \Psi \) obey:

\[
\frac{\delta S[\Psi]}{\delta \Psi} = 0 \quad (\ast)
\]

These will be the equations of motion for the fields \( \Psi \).

**Definition of (\ast)?**

**Def:** A "variation \( S[\Psi] \)" of the fields \( \Psi_0(x) \) in a region \( B \) is a one-parameter deformation, \( \Psi_0(\lambda, p) \), with \( \lambda \in (-\epsilon, \epsilon) \), deformation parameter.
so that \( \lambda = 0 \) is non-deformation

1) \( \Psi_{(i)}(0, \rho) = \Psi_{(i)}(\rho) \quad \forall \rho \in M \)

2) \( \Psi_{(i)}(\lambda, \rho) = \Psi_{(i)}(\rho) \quad \forall \lambda, \rho \in M - B \)

i.e. no deformation at all outside region \( B \).

**Def:** Then, we define:

\[
\delta \Psi_{(i)}(\rho) := \left. \frac{\partial \Psi_{(i)}(\lambda, \rho)}{\partial \lambda} \right|_{\lambda = 0}
\]

**Def:** The action principle now reads:

\[
0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right|_{\lambda = 0} \quad \text{for all variations} \ \delta \Psi_{(i)}.
\]

**Evaluate:**

\[
0 = \left. \frac{\partial S}{\partial \lambda} \right|_{\lambda = 0} = \sum_i \int_B \left[ \frac{\partial L}{\partial \Psi_{(i)}^a \cdots} \delta \Psi_{(i)}^{a \cdots} + \frac{\partial L}{\partial \varepsilon_{(i)}^a \cdots} \delta \varepsilon_{(i)}^{a \cdots} \right] \sqrt{g} \, d^4x
\]

**Term I**

**Term II**

by assumption, \( L \) depends also on the 1st cov. derivatives.

Evaluate terms I, II separately:
**Term II:**

We notice:

\[ \delta (\gamma_{(i) a...b}^c d...e}) = (\delta \gamma_{(i) a...b}^c d...e}) j e \]

\[ \Rightarrow \text{Term II} = \sum_i \int_{B} \frac{\partial}{\partial \gamma_{(i) a...b}^c d...e}} \left( \delta \gamma_{(i) a...b}^c d...e} \right) j e \sqrt{g} d^4 x \]

\[ = \sum_i \int_{B} \left[ \left( \frac{\partial}{\partial \gamma_{(i) a...b}^c d...e}} \right) \delta \gamma_{(i) a...b}^c d...e} \right) j e \sqrt{g} d^4 x \]

One term is a "boundary term":

\[ \sum_i \int_{\partial B} K^e j e \sqrt{g} d^4 x \]

\[ = \sum_i \int_{\partial B} \text{div}_\partial K \]

*Can's theorem* \[ \Rightarrow \]

\[ \sum_i \int_{\partial B} \delta_\partial \Omega \]

\[ \text{inner derivative} \] (Recall: \( \text{div}_\partial K = L_{K \Omega} \))

\[ = (\text{d} \circ \delta_\partial) \Omega = \text{d} \Omega \]

**Exercise:**

Show that for all \( \delta \gamma \):

\[ \delta_\partial \Omega = \text{div}_\partial K \]

but: \( K \delta \gamma \) and \( \delta \gamma(p) = 0 \) if \( p \in \partial B \)

by property 2) of variations.

\[ \Rightarrow = 0 ! \]
Thus, term II simplifies and we obtain:

\[ 0 = \left. \frac{\partial S}{\partial \lambda} \right|_{\lambda=0} = \sum_i \left[ \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{\alpha \beta \ldots}} - \left( \frac{\partial \mathcal{L}}{\partial \delta \psi_{(i)}^{\alpha \beta \ldots}} \right) \frac{\delta \psi_{(i)}^{\alpha \beta \ldots}}{\delta \lambda} \right] \nu^i \, d^4x \]

Since must hold for all variations \( \delta \psi \)

\[ \Rightarrow \]

\[ \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{\alpha \beta \ldots}} - \left( \frac{\partial \mathcal{L}}{\partial \psi_{(i)}^{\alpha \beta \ldots}} \right) \frac{\delta \psi_{(i)}^{\alpha \beta \ldots}}{\delta \lambda} = 0 \]

"Euler-Lagrange equations"

Given \( L(\psi) \), these eqns yield the eqns. of motion for \( \psi \).

**Example:** A real-valued scalar field \( \psi \)

- Such \( \psi \) describe e.g.:
  - \( \pi^0 \) meson (quark + anti-quark)
  - inflaton

- **Lagrangian?**
  - Choose geodesic paths at orb. point
  - and appeal to equiv. principle.
  - Obtain from spec. relativ. Lagrangian:

\[ L = -\frac{1}{2} \left( \psi_{;a} \psi_{;b} g^{ab} + \frac{m^2}{\alpha^2} \psi^2 \right) \]

- Euler-Lagrange equation: \( \text{Klein-Gordon equation} \)

\[ \psi_{;ab} g^{ab} - \frac{m^2}{\alpha^2} \psi = 0 \]

(Exercise: verify)
**Example:** The electromagnetic fields

- Assume there are no charges (i.e., there are only EM waves).
- Define the "EM 4-potential" as a real-number-valued one-form $A$.
- Consider the field strength tensor $F$:
  \[ F = dA \]
- Recall that the $E$ and $B$ fields are components of the 2-form $F$ \((\text{up to a factor of } 2)\).

- The Lagrangian (from eqmiv. principle):
  \[ L = -\frac{1}{16\pi} F_{ab} F^{cd} g^{ac} g^{bd} \]
  \((\text{Exercise: write in terms of forms})\)

- Varying w. r. t. $A$, the E.L. equations read:
  \[ F_{abc} g^{bc} = 0 \]
  \(\text{recall: this is } \delta F = 0\)

- It is also true that
  \[ F_{abc} + F_{cabo} + F_{bca} = 0 \]
  "Maxwell eqns."
  \(\text{but this is not an Euler-Lagrange eqn. It is simply: } dF = 0\)
  \(\text{which holds because } F = dA \text{ and } d^2 = 0\)
Example: A charged scalar field $\psi$, complex-valued
(such $\psi$ describe, e.g., $\pi^\pm$ mesons)
together with electromagnetism.

Equiv. principle yields from spec. relativity:

\[
L = -\frac{i}{2} \left( \psi_a^* - i e A_a \psi^b \right) \left( \psi_b^* + i e A_b \psi^a \right) g^{ab}
\]

\[
\frac{-1}{2} m^2 \psi^* \psi - \frac{1}{16 \pi} F_{ab} F^{cd} g^{ac} g^{bd}
\]

Why $\psi$ complex?

Mixed term is linear for $\psi$
If $\psi$ was real, it would be absent:

\[
\begin{align*}
-\frac{i}{2} e A_a \psi^a & \psi^b \\
+ i e A_b & \psi^b \\
= & e A_a \psi^a (\psi^b - i \mu \psi^b) \\
= & 0 \quad \text{if } \psi = \psi^*
\end{align*}
\]

Vary w. resp. to $\psi^* \Rightarrow$ E. L. eqn:

\[
\psi_{ab} g^{ab} - \frac{m^2}{\hbar^2} \psi + i e A_a g^{ab} (\psi_b + i e A_b \psi) + i e A_{ab} g^{ab} \psi = 0
\]

Uhlenbeck part \hspace{1cm} $\psi$ is affected by $A$

and varying w. resp. to $\psi$ yields the comp. cong. equation.

Vary w. resp. to $A_a \Rightarrow$ E. L. eqn:

\[
\frac{1}{4 \pi} F_{abc} g^{bc} - i e \psi (\psi_a^* - i e A_a \psi^b) + i e \psi^* (\psi_a + i e A_a \psi) = 0
\]

Plain Maxwell \hspace{1cm} $A$ is affected by $\psi, \psi^*$.\hspace{1cm} part
Divac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that \( h = 1 \))

\[
(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m) \psi(x) = 0
\]

"Dirac equation"

(D)

where \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \) is a "Spinor"

describes spin \( \frac{1}{2} \) particles such as electrons and quarks

and the four 4x4 matrices \( \gamma^\mu \) obey:

\[
\gamma^\mu \gamma^\nu \gamma^\mu \gamma^\nu = 2 \gamma^\nu
\]

\((\gamma^\nu)^* = (\gamma^\nu)^T \)

Why (D)? Equation (D) is specifically chosen so that each component of \( \psi \) obeys the Klein Gordon equation. Indeed:

\[
(D) \Rightarrow (-i \gamma^\mu \partial_\mu - m)(i \gamma^\nu \partial_\nu - m) \psi = 0
\]

\[
\Rightarrow (+ \gamma^\mu \gamma^\nu \partial_\nu \partial_\mu + i \gamma^\mu \partial_\mu m - im \gamma^\mu \partial_\mu + m^2) \psi = 0
\]

\[\text{symmetric under } \gamma \mapsto \gamma^* \]

\[
\Rightarrow (\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu + m^2) \psi = 0
\]

\[\text{and symmetric part not needed, it would drop out.} \]

\[
\Rightarrow (\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu + m^2) \psi = 0
\]

\[(\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu + m^2) \psi = 0 \]

\((\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu + m^2) \psi = 0 \]

which is the Klein Gordon equation in flat space.
In general relativity:

- By choosing an orthonormal basis, $\{\mathbf{e}_i\}$, we achieve $g^\mu_\nu = \mathbf{e}_\mu \cdot \mathbf{e}_\nu \quad \forall \mu, \nu \in \mathcal{M}$
  i.e. one set of matrices $g^\mu_\nu$ obeying $g^\mu_\nu g^\nu_\rho + g^\rho_\nu g^\nu_\mu = 2g^\mu_\rho$ suffices.
- This motivates:
  \[(i g^\mu_\nu \mathbf{e}_\mu - m) \psi = 0\]
- But what is the covariant derivative of a spinor?
  \[\nabla_{\mathbf{e}_\rho} \psi = ?\]

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a spinor should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector $e_3$ in direction $e_\rho$:

$e_3 \rightarrow e_3 + \nabla_{e_\rho} e_3 = e_3 + \omega^\rho_\sigma(e_\sigma) e_\rho$

Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e., they can at most "rotate" and in 3+1 dim, this is Lorentz transformations.

This is an infinitesimal Lorentz transformation $\Lambda^\rho_\sigma$:

$e_3 \rightarrow \Lambda^\rho_\sigma e_3 \quad \text{with} \quad \Lambda^\rho_\sigma = \delta^\rho_\sigma + \omega^\rho_\sigma(e_\rho)$

because $\omega^\rho_\sigma$ obeys: $\omega^\sigma_\rho = -\omega^\rho_\sigma$ (Which is the defining equation for infinitesimal Lorentz transformations).
Now that we know the infinitesimal Lorentz transformations for any infinitesimal parallel transport:

**Strategy:** Apply the same infinitesimal Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

1. **Assume** $\{\hat{s}_1, \hat{s}_2, \hat{s}_3\}$ are ON basis in Spinor space, i.e.,

$$\Psi = \psi^i(x) \hat{s}_i$$

2. **How do the** $\hat{s}_i$ **transform under Lorentz transformations?**

i.e., what is $\nabla_\nu \hat{s}_i$? (In analogy to $\nabla_\nu \epsilon_\mu = \omega^\mu_{\nu} \epsilon_\mu$)

From special relativity it is known that under infinitesimal Lorentz transformations,

$$\nabla_\nu = \delta_\nu^\mu + \omega^\mu_{\nu}$$

vectors transform as

$$e_\mu \rightarrow e_\mu + \omega^\alpha_{\nu} e_\alpha$$

and the Dirac spinors transform as:

$$\hat{s}_i \rightarrow \hat{s}_i - \frac{i}{4} \omega^\mu_{\nu} [\gamma^\mu, \gamma^\nu] \hat{s}_i$$

Under infinitesimal Lorentz twist, the spinor states by this amount.
Apply to GR:

If a vector $e^\nu$ is infinitesimally parallel transported in the direction of $e_\alpha$, then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\nu_\mu(e_\alpha)$$

which is the value of the connection 1-form, i.e.: local values of the connection form

$$e_\mu \rightarrow e_\mu + \omega^\nu_\mu(e_\alpha)e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

$$\nabla^\nu e_\mu = \omega^\nu_\mu(e_\alpha)e_\nu$$

Now, when a spinor $s_\mu$ is infinitesimally parallel transported in the direction of $e_\alpha$, then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation

$$\omega^\mu_\nu(e_\alpha)$$

which is the value of the connection 1-form. Thus:

local infinitesimal Lorentz transformation, local values of the connection 1-form

$$s_\mu \rightarrow s_\mu - \frac{1}{4} \omega(e_\alpha)_\rho^\nu [\gamma^\rho \gamma_\nu] s_\mu$$

Since, under infinitesimal parallel transport:

$$s_\mu \rightarrow s_\mu + \nabla_\mu s_\mu$$

→ to be determined
The covariant derivative of the basis vectors $\tilde{e}_a$ of Dirac spinors is:

$$\nabla_{\tilde{e}_a} s_i = -\frac{i}{4} \omega_{\mu}^\rho (e_a)_{\mu} [\gamma^\rho, \gamma_i] s_i.$$  

For general Dirac spinors $\Psi(x) = \Psi^i(x) s_i$, the Leibniz rule for $\nabla$ yields:

$$\nabla_{\tilde{e}_a} \Psi = \nabla_{\tilde{e}_a} (\Psi^i(x) s_i) = (\nabla_{\tilde{e}_a} \Psi^i(x)) s_i + \Psi^i(x) \nabla_{\tilde{e}_a} s_i;$$

i.e.:

$$\nabla_{\tilde{e}_a} \Psi = e_a(\Psi) - \frac{i}{4} \omega(e_a)_\mu [\gamma^\mu, \gamma_i] \Psi$$

$e_a(\Psi) = s_i e_a(\Psi)$ vector field

**Dirac equation:**

The general relativistic Dirac equation:

$$(i\gamma^\rho \nabla_{\tilde{e}_\rho} - m) \Psi = 0$$

now takes this explicit form:

$$i\gamma^\rho e_\rho(\Psi) - \frac{i}{4} \omega(e_a)_\mu [\gamma^\mu, \gamma_i] \Psi - m \Psi = 0$$

in a chart, this becomes a directional derivative of $\Psi$.

**Remark:** The relationship between the Dirac operator $D = i\gamma^\rho \nabla_{\tilde{e}_\rho}$ and the Laplace or d’Alembert operator $\Delta$ also becomes:

$$D = d + \delta$$

To this end, one re-interprets the Grassmann algebra of differential forms as a so-called Clifford algebra.