Friedmann-Lemaître cosmological solutions

Experimental evidence:

- The universe is and has been expanding.
- It appears to be spatially essentially isotropic and homogeneous on scales larger than a few (3-4) billion light years.
  (see e.g. Sloan Digital Sky Survey (SDSS) at www.sdss.org)

Idealizing models:

- Assume perfect spatial isotropy and homogeneity:
- $\Rightarrow$ "Friedmann & Lemaître" (later Robertson & Walker) spacetimes

Concretely:

We assume we can model spacetime as a manifold $(M,g)$ with:

$$ M = \mathbb{I} \times \Sigma $$

$$ g = -dt^2 + a^2(t)\, \bar{g} $$

Here:

- $\mathbb{I}$ is an interval, $\mathbb{I} \subset \mathbb{R}$, and $t \in \mathbb{I}$ is called "cosmic time". $a(t)$ is called the "scale factor".
- $(\Sigma, \bar{g})$ is a fully isotropic & homogeneous Riemannian manifold, i.e., a manifold of constant curvature, providing a 3-dim. surface of homogeneity at each point in cosmic time.
What are the possible Riemannian manifolds of constant curvature?

- The Riemannian tensor $\overline{R}_{ijk}$ must be expressible in terms of a constant, say $K$, which fixes the curvature's strength, and the tensorial part can only depend on the metric $\overline{g}$.

$\Rightarrow$ Given the index symmetries of $\overline{R}_{ijk}$, it should (and does) take the form:

$$\overline{R}_{ijk} = K (\overline{g}_{ik} \overline{g}_{je} - \overline{g}_{ie} \overline{g}_{jk})$$

$\Rightarrow \overline{R}_{je} = 2K \overline{g}_{je}, \overline{R} = 6K$

$\Rightarrow$ Using a "Tried" $\Omega^3$:

$$
\overline{R}_{ij} \equiv \frac{1}{2} \overline{R}_{ijk} \overline{\Theta}^k \wedge \overline{\Theta}^l \equiv K \overline{\Theta}^i \wedge \overline{\Theta}^j
$$

(On bases of $T_p(S^3)$, $\forall p$

Role of the signature of $K$:

$K > 0$: $\Rightarrow \Sigma$ is a 3-dim. sphere (that can be embedded e.g. in a 4-dim euclidean (i.e. flat) space: closed universe

$K = 0$: $\Rightarrow \Sigma$ is euclidean $\mathbb{R}^3$, flat, infinite universe

$K < 0$: $\Rightarrow \Sigma$ is a 3-dim. "pseudo-sphere". The constant negative curvature means it is everywhere "like a saddle". These $\Sigma$ also possess $\infty$ volume.

Note: 4-dim pseudo-Riemannian manifolds with constant curvature are called the hyperbolic manifolds.

- Note: $\overline{R}$ and therefore $K$ have units $\frac{1}{(\text{length})^2}$. Thus, by suitable choice of unit of length, we can choose units of length so that: (This is usually done in cosmology)

$$K = -1, 0 \text{ or } 1$$
A tetrad for spacetime:  \( g = -dt^2 + a^2(t) \bar{g} \)

Define a convenient tetrad, i.e., ON basis of each \( T_p(M) \):

- \( \Theta^0 := dt \) with \( t = \text{cosmic time of above} \)
- \( \Theta^i := a(t) \overline{\Theta}^i \) with \( \overline{\Theta}^i \) being the tetr of \( \Sigma \)

Note: The \( \overline{\Theta}^i \) were chosen ON with respect to \( \bar{g} \).

The \( \Theta^i \) are ON with respect to \( g \).

We then have, e.g.:

\[ 1 \text{st structure equation on } \Sigma: \quad j^{(i,j)}_{\mu} = 0 \quad (\Sigma 1) \]

\[ 1 \text{st structure equation on } M: \quad j^{(\mu, \nu)}_{\tau} = 0 \quad (M 1) \]

Determine the 4-connection \( \omega^\nu_\mu \): \( \text{in spatially isotropic homogeneous case} \)

Strategy: Calculate \( d\Theta^i \) in two ways:

1. \[ d\Theta^i = d(a \overline{\Theta}^i) = (da) \Lambda \overline{\Theta}^i + a d\overline{\Theta}^i \]

   \[ = (\frac{da}{dt} dt) \Lambda \overline{\Theta}^i - a \overline{\omega}_j^i \Lambda \overline{\Theta}^j \]

   (use \( a \overline{\Theta}^i = \Theta^i \)) \[ = \frac{a}{a'} \Theta^i \Lambda \Theta^0 - \omega^i_j \Lambda \Theta^j \]

   (use \( \overline{\Theta}^i = \frac{1}{a'} \Theta^i \)) \[ = -\frac{a}{a'} \Theta^i \Lambda \Theta^0 - \omega^i_j \Lambda \Theta^j \]

   (and \( \Theta^0 \Lambda \Theta^0 = -\Theta^i \Lambda \Theta^i \)) \[ = \frac{1}{a'} \omega^i_j \Lambda \Theta^j \]

2. \[ d\Theta^i = \frac{a}{a'} \frac{d\Theta^i}{dt} \]

   \[ \Rightarrow \omega^i_0 = \frac{a}{a'} \Theta^i \quad \text{and} \quad \omega^i_j = \overline{\omega}^i_j \]

\[ \text{(Intention: expansion in normal affine)} \]

(8st)

\[ \text{(Connection between speed and time)} \]

\[ \omega^i_0 = \frac{a}{a'} \Theta^i \quad \text{and} \quad \omega^i_j = \overline{\omega}^i_j \]
The curvature 2-form:

Recall: 2nd structure equations: (analogous to \( \pi_\nu^\mu = \rho \pi^{\rho \pi + \sigma \sigma} \))

\[
\Omega^\nu_\nu = d \omega^\nu_\nu + \omega^\nu_\sigma \wedge \omega^\sigma_\nu \quad (M2)
\]

\[
\overline{\Omega}^i_j = d \overline{\omega}^i_j + \overline{\omega}^i_k \wedge \overline{\omega}^k_j \quad (\Sigma 2)
\]

For \( i, j \in \{1, 2, 3\} \) (afterwards we will calculate \( \Omega^0_i, \Omega^0_0 \))

\[
\Rightarrow \quad \Omega^i_j = d \omega^i_j + \omega^i_k \wedge \omega^k_j \quad \text{(use Box)} \Rightarrow
\]

\[
= d \overline{\omega}^i_j + \overline{\omega}^i_k \wedge \overline{\omega}^k_j + \omega^i_0 \wedge \omega^0_j
\]

\[
\overline{\Omega}^i_j = k \overline{\omega}^i_j \wedge \overline{\omega}^0_0 \quad \text{(It was a consequence of spatial isotropy & homogeneity)}
\]

Recall also:

\[
\overline{\Omega}^i_j = k \overline{\omega}^i_j \wedge \overline{\omega}^0_0 = \frac{k}{a^2} \theta^i \wedge \theta^0
\]

\[
\Rightarrow \quad \Omega^i_j = \frac{k}{a^2} \theta^i \wedge \theta^0 + \frac{\dot{a}}{a^3} \theta^i \wedge \theta^0
\]

\[
= \frac{k + \dot{a}}{a^2} \theta^i \wedge \theta^0
\]

Similarly, one calculates: Exercise: check

\[
\Omega^0_i = \frac{\dot{a}}{a} \theta^0 \wedge \theta^i
\]

Calculate the Einstein tensor:

Recall:

\[
\Omega_{\rho \sigma} = \frac{1}{2} R_{\rho \sigma \epsilon \delta} \theta^\epsilon \wedge \theta^\delta
\]

\[
\Rightarrow \quad \text{We can read off } R_{\rho \sigma \epsilon \delta}.
\]
We obtain the Ricci tensor $R_{\mu\nu}$ and the curvature scalar $R$.

We obtain the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

**Result:**

$$G_{00} = 3 \left( \frac{a^2}{a^2} + \frac{\kappa}{a^2} \right)$$

$$G_{ii} = -2 \frac{\dot{a}}{a} - \frac{a^2}{a^2} - \frac{\kappa}{a^2}$$

$$G_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

i.e., $G_{\mu\nu}$ is diagonal in this frame.

The energy-momentum tensor:

- From $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ we obtain that $T_{\mu\nu}$ must also be diagonal.

- Recall the interpretation of the entries of a diagonal $T_{\mu\nu}$ in terms of matter energy density $\Sigma$, matter pressure $p$ and cosmological constant $\Lambda$ at the origin of geodesic coordinates:

$$T_{\mu\nu} = \begin{pmatrix} \Sigma & p & 0 \\ 0 & p & 0 \\ 0 & 0 & -\frac{1}{8\pi G} \Lambda \end{pmatrix}$$

The only nontrivial dynamics of matter is here its equation of state:

$$\Sigma = \Sigma(p) \quad p = p(\Sigma)$$
What kind of matter causes such a $T_{\mu \nu}$?

**Proposition:**

The $T_{\mu \nu}$ of any F.L. spacetime is always of the form of that of a perfect fluid.

* The matter doesn't have to be fluid - it could also be e.g. a suitable quantum field.

* But the high symmetry of a F.L. spacetime requires that the matter's $T_{\mu \nu}$ matches that of a perfect fluid.

**Proof:** Consider the 4-vector field dual to $\theta^0$:

$$u = \frac{\partial}{\partial t} = e_0,$$

i.e.: $u = u^\mu e_\mu$ with $u^0 = 1$, $u^i = 0$.

Using $u$, $T_{\mu \nu}$ takes the form that characterizes a perfect fluid:

$$T_{\mu \nu} = (\rho + p) u^\mu u_\nu + (p -\Lambda) g_{\mu \nu}$$

Q: If the matter is a fluid, what's the vector field $u$?

Recall:

- $\omega^2 = \frac{1}{2} \sigma$
- $\omega_{\phi} = -\frac{3}{2} \sigma$
- $\omega_{t} = -\frac{1}{2} \sigma$
- $\omega_{\theta} = \frac{1}{2} \sigma$
- $\omega_{\phi} = 0$

Why? $u$ is tangent to timelike geodesics (that stand still in space) (because $u^i e_i = 0$)

$\nabla_u u = \nabla_{e_0} e_0 = \omega^0 (e_0) e_\mu = 0$ (Hodge dual basis)
The Einstein equation:

\[ g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

now consists of merely 2 equations: Exams, exile

\[ g_{00} = 8\pi G T_{00} \Rightarrow \]

\[ 3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = 8\pi G \rho + \Lambda \tag{A} \]

\[ g_{ij} = 8\pi G T_{ij} \Rightarrow \]

\[ -2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G \rho - \Lambda \tag{B} \]

Notice that \( \Lambda \) contributes

- positively to the energy but
- negatively to the pressure.

Observation: \( \frac{k}{a^2} \) occurs in (A) and (B), i.e., we can eliminate it:

\[ -\frac{1}{2} a \left( \text{Eqn}(B) + \frac{1}{3} \text{Eqn}(A) \right) \text{ yields:} \]

\[ \ddot{a} = -\frac{1}{2} a \ 8\pi G \left( \frac{\rho}{3} + \rho \right) - \frac{1}{2} a \Lambda \left( -1 + \frac{1}{3} \right) \]

\[ \Rightarrow \]

\[ \ddot{a} = -\frac{4\pi G}{3} a \left( \rho + 3 \rho \right) + \frac{1}{3} a \Lambda \]

Thus for all \( k \): For ordinary matter must have deceleration, i.e., \( \ddot{a} < 0 \), but a positive cosm. constant \( \Lambda \) can make \( \ddot{a} > 0 \). At present, energy seems to be only sufficiently diluted so that \( \Lambda \) has taken over \( > 70\% \), \( < 30\% \),5 Our gas of galaxies has negligible \( \rho \).
Experimental evidence?

Supernova distance versus brightness data and evidence from cosmic background radiation:

\[ a > 0 \text{ now!} \]

\[ \Rightarrow \text{At present, energy is already sufficiently dilated so that } \Lambda \text{ dominates over } \Omega : \sim 70\%, \Lambda \text{ and } \sim 30\% \text{ of total } \Omega_{\text{dark}}. \]

In the far future, \( \Omega + p \) will have diluted \( \to 0 \), leaving only \( \Lambda \). Then, the Friedmann eqn reads:

\[ 3 \left( \frac{a^2}{a^2} + \frac{k}{a^2} \right) = \Lambda \]

Solutions:

\[ a(t) = \begin{cases} \cosh \left( \frac{\sqrt{2}}{3} t \right) & \text{for } k = 1 \\ \exp \left( \frac{\sqrt{2} t}{3} \right) & \text{for } k = 0 \\ \sinh \left( \frac{\sqrt{2} t}{3} \right) & \text{for } k = -1 \end{cases} \]

\[ \Rightarrow \text{Exponential expansion is predicted!} \]

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General solution strategy with cosm. constant and matter:

- We have 3 unknown functions of time:
  
  \[ a(t), \Omega(t), p(t) \]

  and we have 3 equations that they obey:

  Eqs. A, B and an equation of state \( p = p(\Omega) \) that depends on the "mater":

  \[ p(\Omega) = -\Omega \text{ for pure vacuum energy (e.g., very early universe)} \]

  \[ p(\Omega) = \frac{1}{3} \Omega \text{ for pure radiation (e.g., in the early universe)} \]

  \[ p(\Omega) = 0 \text{ for pure dust (e.g., middle age universe before } \Lambda \text{ took over)} \]

- Observation:

  \[ (\text{Eqn. A}) \]

  \[ \text{The Friedmann eqn. only contains } a, \Omega \text{ but not } p! \]
Idea:

- Try to express $\rho$ as a function of $a$ to obtain $s = s(a)$.
- Using $s(a)$, the Friedmann eqn becomes an ordinary differential equation only for $a(4)$ and we are done!

Indeed, a key equation helps us to find $s(a)$:

**Proposition:** The Einstein eqns A, B, i.e., $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, imply:

$$\frac{d}{da} (s a^3) = -3 \rho a^2 \quad (P)$$

Indeed, when the parameter $w$ in $\rho = w s$ is known, $(P)$ yields $s(a)$:

- For dust, $\rho = 0 \Rightarrow s \sim a^{-3}$
- For radiation, $\rho = s/3 \Rightarrow s \sim a^{-4}$
- For pure $\Lambda: \rho = -s \Rightarrow s = \text{const}$

\[ s \text{ of radiation decays quicker than } s \text{ of dust because radiation is not only diluted, its wavelengths are also stretched, which reduces the energy too.} \]

\[ s \text{ of vacuum energy does not dilute!} \]

Intuitive meaning of $(P)$?

- $(P)$ in the GR version of the continuity equation for adiabatic expansion: $dE = -p dV$
- With $V = a^3$, $E = sV$ yields:
  \[ d(a^3 s) = -p d(a^3) = -3\rho a^2 da \]
- Thus:
  \[ \frac{d}{da} (a^3 s) = -3\rho a^2 \] which is indeed $(P)$. 

Exact proof of proposition (P):

1. The Einstein equation \( G^{\mu \nu} = 8\pi G T^{\mu \nu} \) and \( G^{\mu \nu} = 0 \) imply \( T^{\mu \nu} = 0 \)

2. Here: \( T^{\mu \nu} = (8 + p) u^\mu u^\nu + pg^{\mu \nu} \)

Thus:
\[
0 = T^{\mu \nu}_{\ ; \nu} = \frac{(8_p + p) u^{\mu} u^{\nu} + (8 + p) u^{\mu} u^{\nu} + \rho u g^{\mu \nu}}{(8 + p) u^{\mu} u^{\nu}}
\]

(using \( g_{\mu \nu} = g_{\mu \nu}^{\ ; \nu} \Rightarrow \))
\[
= (\nabla_u \delta + \nabla_p) u^\mu + (8 + p) u^\mu \nabla_u + p^\mu u_{\mu}^\nu
\]

(using \( u_{\mu}^\nu = u_{\mu}^{\ ; \nu} \Rightarrow \))
\[
= -\nabla_u \delta - \nabla_p - (8 + p) \nabla u + u_{\mu} p^\mu
\]

\[
\Rightarrow 0 = \nabla_u \delta + (8 + p)(\nabla u)
\]

(2)

It remains now to calculate \( \nabla_u \delta \):

\[
\nabla \cdot u = \nabla \cdot (u^\lambda \partial^\lambda \delta) = \Theta^\lambda (\nabla \cdot u^\lambda)
\]

Recall:
\[
\Theta^\lambda = \omega^\lambda_{\ ; \nu} e^\nu = \omega^\lambda_{\ ; \nu} (8 + p)_{\ ; \nu}
\]

Recall that:
\[
\omega^\lambda_{\ ; \nu} = \frac{\partial e^\lambda}{\partial t} \Rightarrow
\]
\[
= \frac{d}{dt} \Theta^i (e_i) = 3 \frac{d}{dt}
\]

\( \Rightarrow \) Eqn. (2) becomes:

\[
\dot{\delta} + 3 \frac{d}{dt} (8 + p) = 0
\]

(Recall: \( u = \frac{d}{dt} \))

Thus:

\[
\dot{\delta} \frac{d^2}{dt} (8 + p) a^2 = 0
\]

\[
\frac{d^2}{dt} (8 + p) a^2 + 3(8 + p) a^2 = 0
\]

\[
\frac{d^2}{dt} a^3 + 3(8 + p) a^2 = -3p a^2
\]

\[
\frac{d^2}{dt} a^3 + 3(8 + p) a^2 = -3p a^2
\]

\( \Rightarrow \)

\[
\frac{d}{da} (8a^2) = -3p a^2
\]

This is Eqn (P)