Recall: A key prediction of GR is its own downfall in singularities. Or is it? Can one prove that GR has generic situations that must lead to a singularity, even in the absence of any symmetry?

The plan:
1. Define and study suitable notions of:
   - Causality
   - Horizons (next to define: "Cauchy horizons")
   - Singularities
2. Develop singularity theorems.

Recall: We assume that spacetime is stably causal.

Intuition: Therefore, inextendible paths either:
- go to $\infty$, or
- end in a singularity

$\Rightarrow$ Continue to study inextendible curves
$\Rightarrow$ Arrive at key concepts of Cauchy horizon and global hyperbolicity

Recall:
- We considered the set of points $J^+(S)$ that can somehow be reached from a set $S$, (i.e. the set of points that are affected by $S$)
- Now consider set of points that can only be reached from $S$: (i.e. the set of events that depend on $S$ and only $S$)
**Definition:**

Assume $S \subseteq M$ is a closed achronal set. Then, the "future domain of dependence of $S$" is defined as:

$$D^+(S') := \{ p \in M \mid \text{Every past inextendible causal curve through } p \text{ intersects } S \}$$

**Example:**

Why $q \notin D^+(S)$? Some of its past inextendible causal curves do not intersect $S$ because they get stuck at the hole!

($q$ is affected by events in the "shadow" of the singularity)

**Definition:**

Analogously, the "past domain of dependence of $S$" is:

$$D^-(S') := \{ p \in M \mid \text{Every future inextendible causal curve through } p \text{ intersects } S \}$$

(The set of events $p$ that affect only $S$)

**Definition:**

The "full domain of dependence of $S$" is:

$$D(S) := D^+(S') \cup D^-(S')$$

**Definition:**

The "future Cauchy horizon of $S$", denoted $H^+(S)$, is:

$$H^+(S') := \overline{D^+(S') - I^ -(D^+(S'))}$$

(Nb: $H^+(S)$ is achronal)
Example:

![Diagram](image)

**Definition:**

The "past Cauchy horizon of \( S' \)", denoted \( H^-(S) \), is defined as:

\[ H^-(S') := \overline{D^-(S)} - \text{I}^+(D^-(S)) \]

(set of earliest events that affect \( S \))

**Definition:**

The "full Cauchy horizon of \( S' \)" is defined as:

\[ H(S') := H^+(S) \cup H^-(S) \]

**Proposition:**

\[ H(S') = \mathcal{D}(S') \]

**Definition:**

A closed, achronal set \( S \) is called a "Cauchy surface", if its full Cauchy horizon vanishes, i.e., if

a) \( H(S) = \emptyset \) or equivalently if

b) \( \mathcal{D}(S) = \emptyset \) or equivalently if

c) \( D(S') = M \)

Note: This follows Wald. The definitions by others are equivalent.
Remarks:

- Cauchy surfaces are important because if the conditions on a Cauchy surface are known, then everything on \( M \) can be predicted and retrodicted.
  
  Note: E.g., anti-de Sitter space has no Cauchy surfaces!

- Since a Cauchy surface is achronal, it can be viewed as an "instant in time".

- The term "surface" is motivated by a theorem:

\[
\text{Every Cauchy surface, } \Sigma, \text{ is a } 3\text{-dimensional } C^0\text{ submanifold of } M.
\]

Definition:

If \((M, g)\) possesses a Cauchy surface then it is called "globally hyperbolic".

Remark: We'll need this notion later for a cosmological singularity theorem.

Proposition:

If \((M, g)\) is globally hyperbolic, then:

- There exists a "global time function \( f \)" so that every surface of constant \( f \) is a Cauchy surface.
- \((M, g)\) is strongly (and therefore also strongly) causal.
Recall: Plan is to study inextendible geodesics in order to detect singularities.

Now: How to identify these geodesics which are inextendible because they end at a singularity in the manifold?

First: Avoid trivial cases when manifold is ending but could be extended.

Definition:

We say that \((M, g)\) is inextendible, if it is not isometric to a proper subset of another spacetime \((M', g')\).

⇒ We will always assume that \((M, g)\) is inextendible.

Definition:

A geodesic which is inextendible but possesses a limit range of its affine parameter is called "incomplete".

Note: This is to exclude inextendible geodesics which keep going to ∞.

Definition:

- We say that \((M, g)\) possesses a "singularity" if it possesses an incomplete geodesic.

⇒ We distinguish singularities of null, spacelike and timelike type.
When going along an incomplete geodesic towards a "singularity", 3 things can happen:

I) A scalar constructed from $R_{\alpha\beta\gamma\delta}$, e.g. $R$, $R^\alpha{}_{\alpha\beta\gamma}$, etc diverges.
   $\Rightarrow$ We say it is a "scalar curvature singularity".

II) In a parallel transported tetrad frame, a scalar component of $R_{\alpha\beta\gamma\delta}$ or its covariant derivatives diverge.
    $\Rightarrow$ We say it is a "parallel-propagated curvature singularity".

III) None of the above. Example: "Conical singularity". (cut out a suitable piece and identify the boundaries of the cut)
    $\Rightarrow$ We say it is a non-curvature singularity.

Fundamental problem:

- In concrete solutions, such as Schwarzschild or FRW cosmologies, curvature singularities are obviously present.
- But these spacetimes are highly symmetric.

Do more realistic, i.e. perturbed spacetimes also show these singularities?
Example:

Spherically symmetric dust shell infall.

In Newton gravity: Use catastrophe theory

⇒ e.g., predict mass density to occur, but not if symmetry perturbed!

In Einstein gravity: Use singularity theorems

⇒ e.g., predict black hole singularity to occur, even if symmetry is perturbed, (if assuming e.g., dominant energy cond., etc.)

or also: postdict a cosmological singularity

Remark:

Singularity theorems ⇒ prediction of singularities is robust.

Thus: If quantum gravity is to resolve singularities, it will have to overcome this robustness!

Strategy for singularity theorems:

a) Focus attention on singularities that can be identified by the existence of incomplete
inextendible timelike (or null) geodesics.

Why? It is clear that these are important singularities because observers travelling such a geodesic have their eignitime bounded above and/or below.
Other singularities? (e.g., singularities identified through incomplete geodesics or singularities identified by some other criterion.)

May well exist in addition but the standard singularity theorems do not attempt to predict them too.

b) Basic idea:

Singulatiries can be in the way of geodesics.

⇒

The presence of singularities interferes with the property of geodesics of being extremal length curves.

c) Recall:

\[ \text{Euler-Lagrange equation} \]

Extremizing curve length \( \Rightarrow \) geodesic equation

The geodesic equation is a differential equation.

Thus:

At least locally, geodesics are paths of extremal length:

- Space-like geodesics are curves of shortest proper distance.
- Time-like geodesics are curves of \text{maximal} proper time (i.e., of maximal eigenvalue).
Why maximal?
If there is a timelike curve between two events $p, q$, then there are timelike curves with shorterproper time: just take a longer path and travel it faster.

\[ \text{(C)} \]

Prove that, even in generic spacetimes:

There always exist curves of maximal length between two events.

\[ \text{(D)} \]

What assumptions are needed?
E.g., the assumption that spacetime is globally hyperbolic suffices.

\[ \text{(E)} \]

Further assume that matter obeys a suitable energy condition, (usually the so-called strong energy condition) and use it to prove that geodesics meet a divergence of a quantity called expansion, $\Theta$, in finite proper time.

$\Rightarrow$ these extremal length curves cannot be geodesics with eigentime larger than a certain finite amount either into the past or future.

\[ \text{(F)} \]

Conclude that there are incomplete geodesics, i.e., that we have a singularity in the past (or future).
A singularity theorem:

Assume that: 1. \((M, g)\) is a globally hyperbolic spacetime

- The energy-momentum tensor of matter obeys the "Strong energy condition":
  \[
  \left( T_{\mu \nu} - \frac{1}{2} T g_{\mu \nu} \right) g^{\mu \nu} \geq 0 \quad \forall \text{ timelike } \xi.
  \]

- There exists a \(C^2\) spacelike Cauchy surface \(\Sigma\), on which the trace of the extrinsic curvature, \(K\), is bounded from above by a negative constant \(C\):
  \[
  K(p) \leq C < 0 \quad \text{for all } p \in \Sigma.
  \]

Then:

No past-directed timelike curve from a spacelike hypersurface \(\Sigma\) can have eignevalue, i.e., proper length, larger than \(\frac{3}{C}\).

J.e.: All past-directed timelike geodesics are incomplete.

⇒ There is a cosmological singularity in the finite past!
Extrinsic curvature?

- The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Intuitively: it is the rate of the expansion of spacetime, more precisely its negative, the rate of contraction.

Thus: Assuming $K(p) \leq \xi < 0$ meant that spacetime has a finite minimum expansion rate everywhere on $\Sigma$. => We'll define expansion below in detail.

The strong energy condition?

Recall: The "weak energy condition":

$$T_{\mu\nu}v^\mu v^\nu \geq 0$$ for all timelike $v$; $g(v,v) < 0$

Meaning? For an observer with unit tangent $v$ the local energy density is: $T_{\mu\nu}v^\mu v^\nu \geq 0$

- The "dominant energy condition":

$$T_{\mu\nu}v^\mu v^\nu \geq 0$$ and $K_p K^p \leq 0$

where $v$ is any timelike vector and $K_p := T_{\mu\nu}v^\mu v^\nu$

Meaning? The local energy-momentum flow vector $K$ may not be conserved but has to be non-space-like: flow should be into the future needed for causality.
The "strong energy condition"

Matter is said to obey the strong energy condition iff:

\[
(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) g^{\mu\nu} \geq 0 \quad \forall \text{ timelike } \xi.
\]

as we will discuss below

- **Intuition?** Excludes matter that causes accelerated expansion.
- **Plausible?** Yes, obeyed by known matter. (but not by dark energy)
- **Relationship?** Independent of weak and dominant energy condition.

Concretely: For known matter, \( T_{\mu\nu} \) is diagonalizable to obtain:

\[
T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & p_i \end{pmatrix}
\]

The energy conditions then read:

- **Weak**: \( \rho > 0 \) and \( \rho + p_i > 0 \) for \( i \in \{1,2,3\} \)
- **Dominant**: \( \rho > |p_i| \) for \( i \in \{1,2,3\} \)

**Exercise:** Show this.

**Strong**: \( \rho + \frac{2}{3} p_i \geq 0 \) and \( \rho + p_i \geq 0 \) for \( i \in \{1,2,3\} \)

**Recall**: A cosmological constant \( \Lambda \) can be viewed as a contribution to \( T_{\mu\nu} \).

Indeed, there is no long energy condition for \( \Lambda \), if \( \Lambda = -\frac{1}{3} \Lambda G \).

**Exercise**: Show that the strong energy condition is violated in cosmology if \( w < -\frac{1}{3} \), i.e., if the expansion is accelerating: \( a(t) > 0 \).
Essence of point e):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, \( \Theta \), in finite proper time:

The "expansion", \( \Theta \):

- Consider a "congruence of timelike geodesics" through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \). If parametrized by proper time, their tangent vector field \( \xi \), namely

\[ \xi := \frac{d}{d\tau} \text{proper time} \]

will obey: \( g(\xi, \xi) = -1 \) \( \forall p \).

- Consider now a one-parameter sub-family of these geodesics:

\( \gamma(0, s) \) parametric family of neighboring geodesics.

- a "connecting vector field" 

Then, we define the deviation vector:

\[ \eta := \frac{d}{ds} \]

\( \xi \) a line of constant \( \tau \) value

\( \eta \) a geodesic, i.e., a line of constant \( s \) value
How does $\eta$ change along a geodesic?

$\xi$, $\zeta$ are Riemann normal coordinates for a geodesic traveller.

$$\Rightarrow \frac{d}{dt} \frac{d}{ds} = \frac{d}{ds} \frac{d}{dt}, \quad \text{i.e.,} \quad \left[ \xi, \eta \right] = 0$$

Since the torsion vanishes: $0 = \mathcal{T}(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \left[ \xi, \eta \right]$

$$\Rightarrow \quad \nabla_\xi \eta = \nabla_\eta \xi$$

$$\Rightarrow \quad \xi^\rho \nabla_\rho \eta^\alpha = \eta^\rho \nabla_\rho \xi^\alpha$$

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For $B^\rho_\mu := \xi^\rho \nabla_\rho \eta^\mu$,  

Along the geodesic's direction, $\xi$, the deviation vector $\eta^\mu$ changes its direction and length by $B^\rho_\mu \eta^\mu$.

The tensor $B^\rho_\mu$ can be decomposed covariantly and uniquely into:

\[ B_{\mu \nu} = \omega_{\mu \nu} + \xi_{\mu \nu} + \epsilon_{\nu \mu} \]

(all 3 terms are tensors because the split is covariant)

Cosmic ballet tensor field.  

We have: $\omega_{\mu \nu} = \frac{1}{2} (B_{\mu \nu} - B_{\nu \mu})$, clearly.

But $\epsilon_{\mu \nu}, \xi_{\mu \nu} = ?$

\[ B_{\mu \nu} = \omega_{\mu \nu} + \xi_{\mu \nu} + \epsilon_{\nu \mu} \]

In preparation: define the projector $h_{\mu \nu}$ onto $(\mathcal{R} \xi)^{-1}$, i.e. onto the spatial components:

\[ h_{\mu \nu} := g_{\mu \nu} + \xi_{\mu} \xi_{\nu} \]

Check: is $h_{\mu \nu} \xi^\nu$ really always $\perp$ to $\xi$?

Indeed: $\xi^\mu h_{\mu \nu} \xi^\nu = (\xi, \pi) + (\xi, \xi)(\xi, \pi) = 0$
Define: The "expansion", \( \Theta \), is defined as the magnitude of the spatial part of \( B \):

\[
\Theta := B^{\mu\nu} h_{\mu\nu}
\]

Claim: \( \text{Tr}(B) = \Theta \)

Indeed:

\[
\Theta = B^{\mu\nu} h_{\mu\nu} = B^{\mu\nu} g_{\mu\nu} + \delta^\mu_\nu \delta^\nu_\mu B_{\mu\nu} = \text{Tr}(B) + \frac{\Theta}{3} \delta^\mu_\nu h_{\mu\nu}
\]

\((=0 \text{ because } \eta_{\mu\nu} = 0)\)

Therefore:

\[
\sigma_{\mu\nu} = \frac{1}{3} \left( B_{\mu\nu} + B_{\nu\mu} \right) - \frac{1}{3} \Theta h_{\mu\nu}
\]

because:

\[
\text{Tr}(B) = \frac{1}{2} \left( g_{\mu\nu} + g_{\nu\mu} \right) = \frac{1}{2} \left( g_{\mu\nu} + g_{\mu\nu} \right)
\]

However:

\[
\text{the part of } B_{\mu\nu} \text{ which is symmetric and traceless}
\]

and:

\[
\tau_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} \quad \text{is the "rest term".}
\]

Interpretation:

a) \( \omega_{\mu\nu} \) is antisymmetric: \( \omega_{\mu\nu} = -\omega_{\nu\mu} \)

\( \Rightarrow \) it generates Lorentz transformation for \( \eta \).

but all \( \eta \) are \( \perp \) to the time direction

\( \Rightarrow \) \( \omega_{\mu\nu} \) generates spatial rotations of neighboring geodesics around another. So, \( \omega_{\mu\nu} \) is called \( \omega = "Twists tensor" \)

One can prove: (nontrivial)

If one chooses the congruence of geodesics \( \perp \) to \( \Sigma \), then \( \omega_{\mu\nu} = 0 \).
b) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitian)

Consider "diagonalized", by suitable choice of cd basis.

$\Rightarrow \sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

Note: Since $Tr(\sigma) = 0$ we have $det(e^{t\sigma}) = 1$

$\Rightarrow$ The volume spanned by basis' vectors stays the same under the action of $\sigma$.

$\Rightarrow$ **Definition:** $\sigma_{\mu\nu} = \text{"Shear tensor"}$

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c.) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the twist part, $\Theta$, i.e., more precisely

$$\Theta_{\mu\nu} = \frac{1}{2} \Theta h_{\mu\nu} = \text{"Expansion tensor"}$$

recall: is projector on spatial part.

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\Theta$ along a geodesic?