GR for Cosmology, Achim Kempf, Fall 2015, Lecture 22

Note Title

Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics meet a divergence of a quantity called expansion, θ, in finite proper time in the past and this will mean a big bang singularity:

important notion also e.g. in study of grav. collaps of stars.

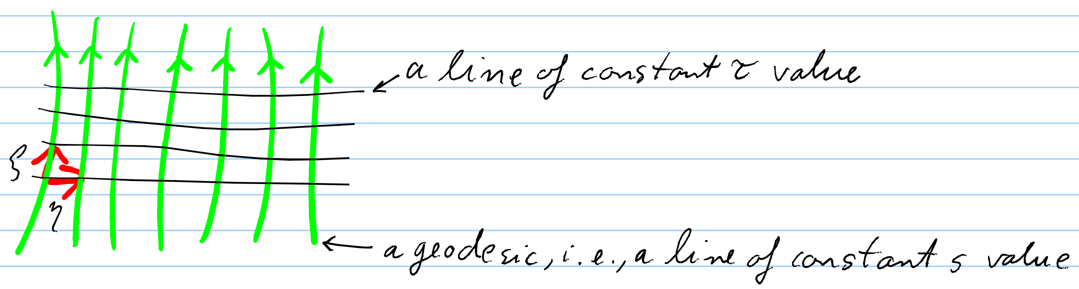
The "expansion", θ:

- Consider a "congruence of timelike geodesics" (e.g., freely falling dust.) through Σ, i.e., a smooth family of timelike geodesics, exactly one through each $p \in \Sigma$: (Σ is a Cauchy surface)

- We consider a one-parameter sub-family of these geodesics:

$$\gamma(\tau, s)$$

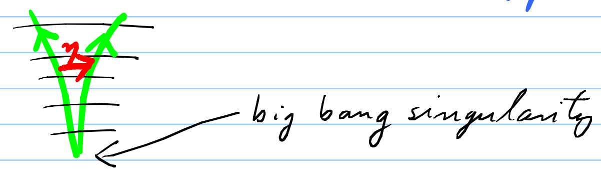
τ: eigentime; s: parameter of family of neighboring geodesics.



- Then, we define the deviation vector to a neighboring geodesic:

$$\eta := \frac{d}{ds}$$

- The singularity theorem claims that this happened in the past:



How does η change along a past-directed timelike geodesic with tangent ξ?

We showed:

$$\boxed{\xi^\mu \eta^\nu{}_{;\mu} = \eta^\alpha B^\nu{}_\alpha \quad \text{where} \quad B^\nu{}_\mu := \xi^\nu{}_{;\mu}}$$

$\Rightarrow$ Along the geodesic, ξ, the deviation vector η^μ changes its direction and length by $B^\nu{}_\mu \eta^\mu$.

- The tensor $B^\nu{}_\mu$ can be decomposed covariantly and uniquely:

$$B_{\mu\nu} = \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} + \overbrace{\sigma_{\mu\nu}}^{\text{symmetric and trace}=0} + \underbrace{t_{\mu\nu}}_{\text{rest}}$$

Explicitly:

Volume preserving:

$$\omega_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} - B_{\nu\mu}\right) \qquad \text{Twist:}$$

$$\sigma_{\mu\nu} = \frac{1}{2}\left(B_{\mu\nu} + B_{\nu\mu}\right) - \frac{1}{3}\theta h_{\mu\nu} \qquad \text{Shear:}$$

Volume changing:

$$t_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} \qquad \text{Expansion:}$$

Here, we defined: $\theta := B^{\mu\nu} g_{\mu\nu}$ and $h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$

I.e., the Expansion, θ, is the trace of B, which we showed is also equal to the magnitude of the spatial part of B: $\theta = B^{\mu\nu} h_{\mu\nu}$.

Key question: $\boxed{\text{What is the dynamics of } \theta \text{?}}$

The Raychaudhuri equation

For the derivation, we will use:

A) Definition of B is: $B_{\mu\nu} := \xi_{\mu;\nu}$

B) The curvature tensor obeys the Ricci equation:

$$\boxed{\xi^a{}_{;bc} - \xi^a{}_{;cb} = R^a{}_{bcd}\,\xi^d}$$

c) ξ is tangent to a geodesic, i.e., it obeys: $\nabla_\xi \xi = 0$

i.e.: $0 = \nabla_{\xi^a e_a} \xi^b e_b = \xi^a \nabla_{e_a} \xi^b e_b = \xi^a \xi^b{}_{;a} e_b$

True for all e_a, thus: $\boxed{\xi^a \xi^b{}_{;a} = 0}$

Now calculate the rate of change of B along the geodesic:

$$\underbrace{\xi^c B_{ab;c}}_{\nabla_\xi B} \overset{(A)}{=} \xi^c \xi_{a;bc}$$

$$\overset{(B)}{=} \xi^c \xi_{a;cb} + \xi^c R_{abcd}\xi^d$$

$$\overset{\text{Leibniz rule}}{=} \underbrace{(\xi^c \xi_{a;c})_{;b}}_{=0} - \xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c \xi^d$$

$$\overset{(C)}{=} -\xi^c{}_{;b}\,\xi_{a;c} + R_{abcd}\,\xi^c\xi^d$$

$$\overset{(A)}{=} -B^c{}_b\, B_{ac} + R_{abcd}\,\xi^c\xi^d$$

In summary, we derived:

$$\xi^c B_{ab;c} = -B^c{}_b B_{ac} + R_{abcd}\,\xi^c \xi^d \qquad (*)$$

The trace of $(*)$ will be the Raychaudhuri equation.

But first, we recall:

- $\xi = \dfrac{d}{d\tau}$

- $\mathrm{Tr}\, B = B_{\mu\nu} g^{\mu\nu} = \theta$

$\Rightarrow$ Trace(LHS) of $(*)$ reads $\dfrac{d}{d\tau}\theta$!

Now on the RHS of $(*)$ use the decomposition

$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}$ to express $B^c{}_b B_{ac}$:

$$\begin{aligned} B^c{}_b B_{ac} &= \omega^c{}_b \left(\underline{\omega_{ac}} + \sigma_{ac} + \tfrac{1}{3}\theta h_{ac}\right) \\ &+ \sigma^c{}_b \left(\omega_{ac} + \underline{\sigma_{ac}} + \tfrac{1}{3}\theta h_{ac}\right) \\ &+ \tfrac{1}{3}\theta h^c{}_b \left(\omega_{ac} + \sigma_{ac} + \underline{\tfrac{1}{3}\theta h_{ac}}\right) \end{aligned}$$

When taking the trace, $g^{ab} B^c{}_b B_{ac}$, only the diagonal terms survive:

Exercise: show it is 3

$$\mathrm{Tr}(BB) = g^{ab} B^c{}_b B_{ac} = \omega_{ab}\omega^{ab} + \sigma_{ab}\sigma^{ab} + \frac{1}{9}\theta^2 \overbrace{h_{ab}h^{ab}}^{3}$$

The Raychaudhuri equation is then the trace of Eq. $(*)$:

recall: Ricci tensor is $R_{cd} = R_{cda}{}^a$

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \underbrace{\sigma_{ab}\sigma^{ab}}_{\text{always positive}} - \underbrace{\omega_{ab}\omega^{ab}}_{\substack{\text{always positive} \\ \text{(and vanishes if} \\ \text{choose congruence } \perp \Sigma)}} - \underbrace{R_{cd}\,\xi^c\xi^d}_{\text{pos. or neg?}}$$

?

Dynamics

Assume that

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for all timelike } \xi$$

i.e., using the Einstein equation

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T^a{}_a\right)$$

we are assuming that

$$T_{\mu\nu}\xi^\mu\xi^\nu - \tfrac{1}{2}\underbrace{\xi^\mu\xi_\mu}_{=-1} T \geq 0 \quad \text{whenever } \xi^\mu\xi_\mu < 0$$

i.e. the **Strong Energy Condition**.

Thus, assuming the strong energy condition:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0$$

i.e., $$-\frac{1}{\theta^2}\frac{d\theta}{d\tau} - \frac{1}{3} \geq 0$$

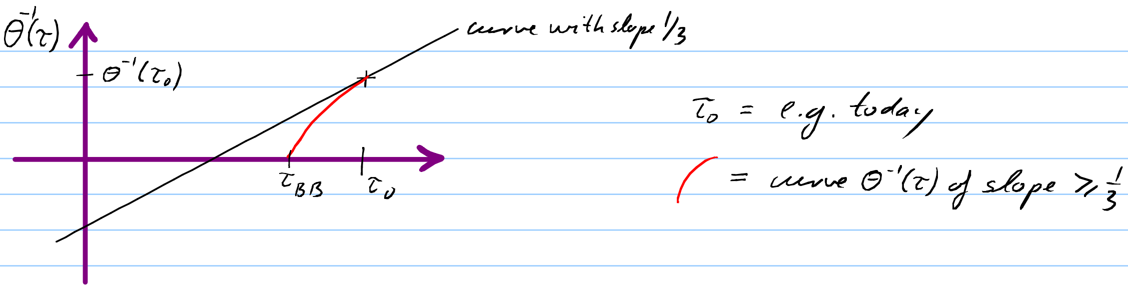
i.e., $$\boxed{\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}} \qquad (\dagger)$$

Consider the cases when the geodesics are initially all either

a.) diverging, i.e., $\theta(\tau_0) > 0$ (expanding universe) or

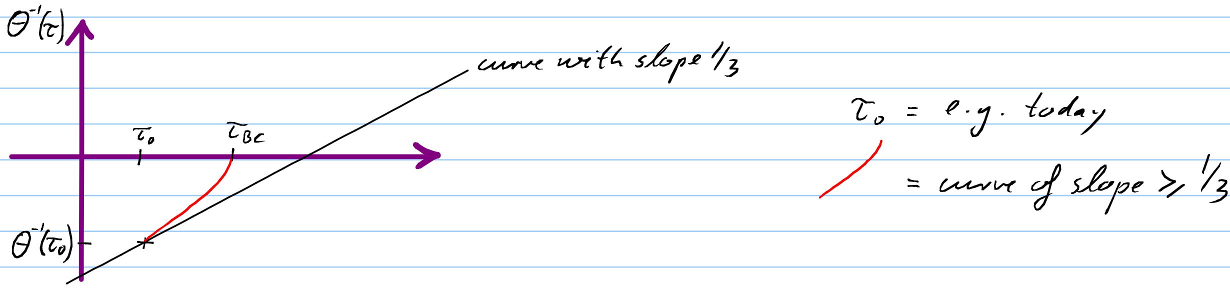
b.) converging, i.e., $\theta(\tau_0) < 0$ (contracting universe)

(This is reformulating the theorem's assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere) a.)



We see that $\theta^{-1}(\tau)$ must have hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BB} (Big Bang).

b)



We see that $\theta^{-1}(\tau)$ will hit $\theta^{-1}(\tau)=0$ at a finite time τ_{BC} (Big Crunch)

Conclusion:

Eq. (+) implies that $\theta^{-1}(\tau)$ must go through 0, i.e.:

a.) for sufficiently early τ, have $\theta \to +\infty$, i.e: Big Bang

b.) for sufficiently late τ, have $\theta \to -\infty$, i.e.: Big Crunch

Note: This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.

Results, e.g., regarding types of cosmic singularity?

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence e.g. of geodesics.

⇒ We can now explicitly calculate the twist, shear and expansion along the congruence:

The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.
- Recall:

  The expansion in one direction can be say enhanced by shear, as long as shear shrinks other directions.
- Definition:

  We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \sigma_{\mu\nu} + \frac{1}{3}\,\theta\, h_{\mu\nu}$$

shear

projector $\perp$ to the timelike u-field

symmetric part of $B_{\mu\nu}$

expansion scalar.

- $\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & \theta_1 & & \\ & & \theta_2 & \\ & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $\underbrace{Tr(h_{\mu\nu})}_{\Rightarrow \text{ is not quite projector}} = 3$

- **Definition:**

$$H_i := \frac{1}{3}\theta_i$$

Local Hubble expansion function in direction e_i.

$$H := \frac{1}{3}\theta$$

Overall local Hubble expansion function.

- **Definition:**

We use H_i, H to define local directional and general scale factors ℓ_i, ℓ:

The ℓ_i, ℓ are defined as the solutions to:

$$\frac{\dot{\ell}_i}{\ell_i} = H_i$$

$$\frac{\dot{\ell}}{\ell} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{\ell} = u(\ell) = u^\mu \frac{\partial}{\partial x^\mu}\ell$$

- What behavior can occur in the far past?

Full set of cases not yet known.

But:

Explicit examples are known where e.g.:

- All $l_i \to 0$ as in FL cosmologies
- $l_1, l_2 \to 0$, $l_3 \to \infty$ "cigar singularity"
- $l_1, l_2 \to 0$, $l_3 \to const$ "barrel singularity"
- $l_1, l_2 \to const$, $l_3 \to 0$ "pancake singularity"

- **Note:** For homogeneous, isotropic FL models, H is the regular Hubble parameter and l is its scale factor.

Regarding black holes:

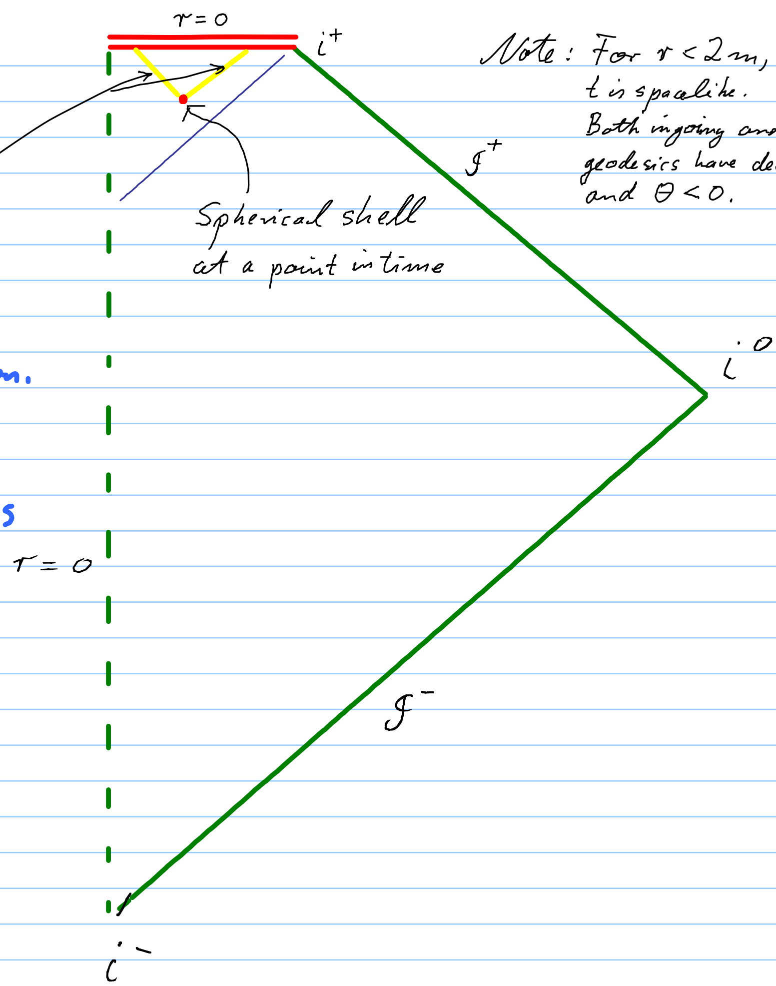
Recall: Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

Def:

- Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T.
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

Examples: All concentric spheres inside a Schwarzschild black hole.

Note: For $r < 2m$, r is timelike and t is spacelike.
Both ingoing and outgoing null geodesics have decreasing r, and $\theta < 0$.



The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.

Def: Let Σ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathfrak{T}$, of all trapped surfaces $T \subset \Sigma$ is called the **trapped region** of Σ.

Def: The boundary $\partial\mathfrak{T} \subset \Sigma$ is called the **apparent horizon** of the spacelike hypersurface Σ.

Note: $\partial\mathfrak{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha, \ \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathfrak{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup_\alpha \mathfrak{T}_\alpha$$

is called the **Trapping horizon** of the spacetime.

Remarks:

- To check for the existence of an event horizon

$$\dot{J}^-(\text{worldline to } i^+)$$

in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- For static Schwarzschild black holes the event and apparent horizons coincide.

- For general black holes, apparent horizons are on or inside the event horizons.