Recall:

$M$ is a manifold and $\mathbb{R}^n$ is the coordinate system. Charts are tools to get a handle at the otherwise nameless abstract points of the manifold.

Problem:

How to define the abstract "Tangent space, $T_p(M)$," to a differentiable manifold at a point $p$?

Intuition:

E.g., a 2-dim manifold has 2-dim vector space of tangent vectors.

⇒ Proper definition should imply:

An $n$-dim manifold possesses for every point $p$ an $n$-dim vector space of tangent vectors.
3 equivalent definitions of $T_p(M)$:

1. "Algebraic" definition of $T_p(M):
   Idea: A tangent vector = directional derivative,
   Derivatives definable through Leibniz rule:
   $$(fg)' = f'g + fg'$$

2. "Physicist" definition of $T_p(M):
   Idea: The elements of $T_p(M)$ are
   to be vectors recognizable by how
   their components change with charts.

3. "Geometric" definition of $T_p(M):
   Idea: The elements of $T_p(M)$ are
   to be actual tangent vectors
   of one-dim. paths in the
   manifold, that pass through p.
The 3 defs are equivalent, but:

We'll need all 3 occasionally!

⇒ we will do all 3:

1. Algebraic definition of $T_p(M)$

   Idea: a) A tangent vector = directional derivative,

   b) Derivatives definable through Leibniz rule:

   $$(fg)' = f'g + fg'$$

Key example: $M = \mathbb{R}^m$

   a) The tangent vectors $\xi$ at a point $p$ are identified with the directional 1st derivatives:

   $$\xi = \sum_{i=1}^{m} \xi_i \frac{2}{\partial x_i} \bigg|_{x=p}$$

   b) Thus, tangent vectors at $p$ should be those maps

   $\xi : f \rightarrow \xi(f) = \sum_{i=1}^{m} \xi_i \frac{2}{\partial x_i} f(x) \bigg|_{x=p}$

   which obey the "Leibniz rule" at $p$:

   $$\xi(fg) = \xi(f)g + f\xi(g) \bigg|_{x=p}$$

Q: How to express the local nature of $\xi \in T_p(M)$ properly?
A: $\xi$ acts on function germs, not on functions.

Def: Assume $M, N$ are differentiable manifolds and $p \in M$.

- We say that two differentiable functions $\phi, \psi$ are germ-equivalent about $p$ if in a neighborhood $U \subset M$ of $p$:
  $$\phi(q) = \psi(q) \quad \forall q \in U$$
- Each such equivalence class of functions is called a germ at $p$.
- Then, the "germ" of $\phi$ at $p$, denoted $\overline{\phi}_p$, is the equivalence class of all functions $\psi$ which are identical to $\phi$ in some neighborhood of $p$:
  $$\psi \in \overline{\phi}_p \iff \exists U_p \forall q \in U_p : \phi(q) = \psi(q)$$

Notice: Assume $\phi : M \to N$ is differentiable at $p \in M$.

Then all $\psi \in \overline{\phi}_p$ possess the same first derivative at $p$.

For example:

Consider germs of scalar functions $f$:

$$M \ni p \xrightarrow{f} \mathbb{R}, \text{ i.e.: } f : M \to \mathbb{R}$$
Note: To specify a germ, it suffices to specify any arbitrary one of its functions. The set of all germs at $p$ is denoted $\mathcal{F}(p)$. 

Note: One has for all $c \in \mathbb{R}$ and $f, g \in \mathcal{F}(p)$:

(a) $c \cdot \overline{f} = \overline{c \cdot f}$
(b) $\overline{f \cdot g} = \overline{f} \cdot \overline{g}$
(c) $\overline{f + g} = \overline{f} + \overline{g}$

$\Rightarrow \mathcal{F}(p)$ obeys the axioms of an associative algebra.

Finally: Algebraic definition of $T_p(M)$

Recall idea: The elements of $T_p(M)$ are to be 1st derivatives $\Rightarrow$ definable by Leibniz rule.

Definition: The tangent space $T_p(M)$ is the set of "derivations" of $\mathcal{F}(p)$, i.e. the set of linear maps $\xi: \mathcal{F}(p) \rightarrow \mathbb{R}$ which obey:

$$\xi\left(\overline{f_p \cdot g_p}\right) = \xi\left(\overline{f_p}\right) \cdot \overline{g_p(p)} + \overline{f_p(p)} \cdot \xi\left(\overline{g_p}\right)$$

Remember this.
Remark: This definition is abstract enough not only for arbitrary differentiable manifolds! This definition (as derivations of the algebra of functions) is also suitable for "Noncommutative Geometry": There, (Quantum Gravity) the algebra of functions \( F(p) \) is noncommutative.

Note: Can’t do Newton’s derivative then but algebraic def’n of derivation still works.

First example: a constant function, \( c \), and its germ \( \bar{c} \).

\[ c(x) := c \quad \text{and } c \text{ is a constant: } c \in \mathbb{R} \]

Then: \( \xi(\bar{c}) = 0 \) for all \( \xi \in T_p(M) \)

Proof: \( \xi(\bar{c}) = c \xi(1) = c \xi(1) \cdot 1 = c(\xi(1) 1 + \xi(1)) \)

\[ = 2c \xi(1) \implies \xi(\bar{c}) = 0 \checkmark \]
Example: The case $M = \mathbb{R}^n$

If our definition for $T_p(M)$ is good, we expect that every $\xi \in T_p(M)$ is of the form:

$$\xi = \sum_{i=1}^{\infty} \xi_i \frac{\partial}{\partial x^i}|_{x=p}$$

Proof:

- We choose $p$ to have coordinates $x = (0,0,\ldots)$.

- Assume $\xi \in T_p(M)$ and $\bar{f} \in \bar{F}(p)$.

Notation: $h_{ij}(a',\ldots,a^n) = \frac{\partial^2}{\partial a_i \partial a_j} h(a',\ldots,a^n)$

Then:

- Note: there are not 3 numbers! These are 3 functions, i.e., 3 equivalence classes of functions.

- \[ \xi(\bar{f}(x)) = \xi(\bar{f}(0) + \bar{f}(x) - \bar{f}(0)) \]

- \[ = \xi(\bar{f}(0)) + \int_0^1 \frac{d}{dt} \bar{F}(tx',\ldots,tx^n) \, dt \]

- \[ = \xi(\bar{f}(0)) + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \bar{F}(tx',\ldots,tx^n)}{\partial x^j} \frac{d(tx^i)}{dt} \, dt \]

- \[ = \xi \left( \int_0^1 \sum_{i=1}^n \bar{F}_i (tx',\ldots,tx^n) x^i \, dt \right) \]
Linearity of $\xi$:

$$
\sum_{i=1}^{n} \xi \left( \int_{0}^{\bar{x}_{i}} f_{i}(tx', \ldots, tx^n) dt \cdot \bar{x}_{i} \right)
$$

Leibniz rule:

$$
\sum_{i=1}^{n} \xi \left( \int_{0}^{\bar{x}_{i}} f_{i}(tx', \ldots, tx^n) dt \right) \cdot \bar{x}_{i} \left|_{x=p=0} \right. \\
+ \sum_{i=1}^{n} \left( \int_{0}^{\bar{x}_{i}} f_{i}(tx', \ldots, tx^n) dt \right) \left. \cdot \xi \left( \bar{x}_{i} \right) \right|_{x=p=0}
$$

$$
= \sum_{i=1}^{n} \xi (\bar{x}_{i}) \int_{0}^{\bar{x}_{i}} f_{i}(0, \ldots, 0) dt
$$

$$
= \sum_{i=1}^{n} \xi (\bar{x}_{i}) \frac{\partial}{\partial x_{i}} f(x', \ldots, x^n) \bigg|_{x=p=0}
$$

$$(\int_{0}^{\bar{x}_{i}} dt = c \Rightarrow)$$

$$
= \sum_{i=1}^{n} \xi (\bar{x}_{i}) \frac{\partial}{\partial x_{i}} f(x', \ldots, x^n) \bigg|_{x=p=0}
$$

Indeed, every $\xi \in \mathcal{T}_{\rho}(\mathcal{M})$ is of the form

$$
\xi = \sum_{i=1}^{n} \xi (\bar{x}_{i}) \frac{\partial}{\partial x_{i}} f(x', \ldots, x^n) \bigg|_{x=p,}
$$

(\text{I})

namely with

$$
\xi' = \xi (\bar{x}_{i})
$$

(\text{II})

Notice: Knowing how $\xi$ acts on the coordinate functions $\bar{x}_{i}$ yields $\xi'$ (from II) and thus it means we know how $\xi$ acts on all functions $f \in \mathcal{F}(\rho)$, namely through (I).
But:

- This was the simple example:
  \[ M = \mathbb{R}^n \]

- How does our definition of \( T_p(M) \) work for \( M \neq \mathbb{R}^n \), concretely?

Recall:

- \( h \) gives abstract points a name, i.e. makes them count.

Problem: How to make abstract \( \xi \in T_p(M) \) count?

Solution: Make use of charts in our way!

Preparation: \( T_p(M) \) and Diffeomorphisms.

Consider two differentiable manifolds, \( M \) and \( N \):

Note: If \( N = \mathbb{R}^n \), then \( \xi \) is a chart.

(that's the case we'll need but it's easy to keep a general \( N \) too)
Here: \( \mathcal{F}(q) \) and \( \mathcal{F}(p) \) are algebra of function (groms).

Given \( \mathcal{E} \) we obtain a map \( \mathcal{E}^*: \mathcal{F}(q) \to \mathcal{F}(p) \)

\[ \mathcal{E}^*: g \mapsto f = \mathcal{E}^*(g) \text{ with } f(x) = g(\mathcal{E}(x)) \quad \forall x \in M \]

i.e.: \( f = \mathcal{E}^*(g) = g \circ \mathcal{E} \quad (+) \)

Here: Given \( \mathcal{E}^*: \mathcal{F}(q) \to \mathcal{F}(p) \) we obtain the "tangent map":

\[ T_\mathcal{E} : T_p(M) \to T_q(M) \]

\[ T_\mathcal{E} : \mathbf{g} \to \mathbf{\eta} \]
Namely: \[ \gamma = \xi \circ \mathcal{E}^+ \]
i.e.: \[ \gamma(g) = \xi(\mathcal{E}^+(g)) \]

From (+) \Rightarrow \[ \gamma(g) = \xi(\mathcal{E} \circ \mathcal{E}^+(g)) \]

The crucial special case:

- \( N = \mathbb{R}^n \) (with \( n = \dim(N) \))
- \( \mathcal{E} \) is invertible 
- \( \mathcal{E} \) is algebra isomorphism 
- \( \therefore T_p \mathcal{E} \) is vector space isomorphism

\[ \Rightarrow \text{We do obtain a concrete handle on the abstract tangent vectors} \; \xi \in T_p(M), \text{ given a chart} \; h \]

Namely:
Given a chart \( \mathcal{C} \), every abstract point \( p \in M \) has a concrete image \( \mathcal{C}(p) \in \mathbb{R}^m \), and:

- Every abstract vector \( \xi \in T_p(M) \) has a concrete image \( \eta \in T_{\mathcal{C}(p)}(\mathbb{R}^m) \)  namely:

  \[ \eta = T_p \mathcal{C}(\xi) \]

- The image \( \eta \) is concrete because \( \eta \) is tangent vector to a point \( q \in \mathbb{R}^m \), and it therefore must take the form (we showed this):

  \[ \eta = \sum_{i=1}^{\infty} \eta_i \frac{\partial}{\partial x^i} \bigg|_{x=q} \]

Conversely: (and very conveniently)

- Assuming a fixed \( \mathcal{C} \), any choice of a \( q = (x',...,x^m) \) denotes a \( p \in M \) and any choice of a \( (q',...,q^n) \) denotes a \( \xi \in T_p(M) \).
E.g. \( \eta = \frac{\partial}{\partial x} \bigg|_{x=q} \) is the image of some abstract \( \xi \in T_p(M) \), for fixed \( \xi \).

Notation: \( \eta = \frac{\partial}{\partial x} \bigg|_{x=p} \)

[Symbolic notation]

Next:

If we hold \( p \) and \( \xi \in T_p(M) \) fixed,

how do the numbers \( (x', \ldots, x^n) \)

and \( (\eta', \ldots, \eta^n) \) change when we

change the chart? \[\text{Physicists' def of } T_p(M)\]