Note Title

GR for Cosmology, Achim Kempf, Fall 2015, Lecture 4

Differential forms (also called "exterior differential forms")

Why? Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p(M)^*$ at p:

- Each $\omega \in T_p(M)^*$ is a lin. map:

$$\omega : T_p(M) \to \mathbb{R}$$
$$\omega : \xi \to \omega(\xi)$$

- Each such ω is a covariant tensor, of rank $(0,1)$

More generally, consider the covariant tensors of rank $(0,r)$:

- Recall: $T_p(M)_r := \underbrace{T_p(M)^* \otimes \ldots \otimes T_p(M)^*}_{r \text{ factors}}$

- Each $v \in T_p(M)_r$ is a multi-linear map:

$$v : T_p(M)^r \to \mathbb{R}$$

- In particular, if $\xi_1, \ldots, \xi_r \in T_p(M)$ then:

$$v : \xi_1 \times \ldots \times \xi_r \to v(\xi_1, \ldots, \xi_r)$$

Definition: If $r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of v" as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A:

$$\tilde{v}(\xi_1, \dots, \xi_r) = A(v)(\xi_1, \dots, \xi_r) := \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, v(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)})$$

the sign (± 1) of the permutation σ

group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

Concretely:

- Consider $v := df \otimes dg$, for $f, g \in \mathfrak{F}_p(M)$
- Then $v(\xi_1, \xi_2) = df(\xi_1)\, dg(\xi_2)$ (which is $= \xi_1(f)\,\xi_2(g)$)
- Apply A:

$$\tilde{v}(\xi_1, \xi_2) = Av(\xi_1, \xi_2) = \frac{1}{2}\left(df(\xi_1)\, dg(\xi_2) - df(\xi_2)\, dg(\xi_1)\right)$$

- $\Rightarrow$ We can also write:

$$A(df \otimes dg) = \frac{1}{2}(df \otimes dg - dg \otimes df)$$

Proposition: A is a projector, i.e., it obeys

$$A \circ A = A$$

Check in above example:

$$A \circ A(df \otimes dg) = A\left(\tfrac{1}{2}\, df \otimes dg - \tfrac{1}{2}\, dg \otimes df\right)$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}\, df \otimes dg - \tfrac{1}{2}\, dg \otimes df\right) - \tfrac{1}{2}\left(\tfrac{1}{2}\, dg \otimes df - \tfrac{1}{2}\, df \otimes dg\right)$$

$$= \tfrac{1}{2}\left(df \otimes dg - dg \otimes df\right)$$

$$= A(df \otimes dg)$$

Definition:

For $r > 1$ we define the space of differential r-forms (or 'exterior' r-forms) $\Lambda_r(p)$ at $p \in M$ as the subspace of totally anti-symmetric tensors of rank $(0, r)$:

a projector on a vector space

a vector space

a vector space

$$\Lambda_r(p) := A\, T_p(M)_r$$

$\rightsquigarrow$ So if $v \in \Lambda_r(p)$ then $\tilde{v} = A(v) = v$

Definition: For $r=0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := \mathbb{R}$$

(for 0 forms on the entire manifold we will have $\Lambda_0 := \mathcal{F}(M)$)

For $r=1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

$$\Lambda_1(p) := T_p(M),$$

Strategy now:

Define multiplication $\rightarrow$ obtain algebra $\rightarrow$ obtain derivations

The wedge product:

Def: If $\omega \in \Lambda_s(p)$, $\nu \in \Lambda_r(p)$, and $r,s \neq 0$ then the wedge product $\wedge$ yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \to \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \to \omega \wedge \nu = \underbrace{\frac{(s+r)!}{s!\,r!}}_{\text{a normalization factor}} A(\omega \otimes \nu)$$

Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\,\omega$

Note: $dx^i \wedge dx^i = 0 \;\forall i$

Example: For dx^i, dx^j we obtain:

$$dx^i \wedge dx^j = \left(dx^i \otimes dx^j - dx^j \otimes dx^i\right)$$

Properties of $\wedge$:

- bi-linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \quad \text{if } a \in \mathbb{R}$$

- associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

- "graded" commutative:

(E.g. for $dx^i \in \Lambda_1(p)$: $dx^i \wedge dx^j = -dx^j \wedge dx^i$ since $r = s = 1$.)

$$\omega \wedge \nu = (-1)^{rs} \, \nu \wedge \omega \quad \text{if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

- We can use $\wedge$ to build bases of $\Lambda_r(p)$:

Assume: $\{\theta^i\}_{i=1}^{n}$ is a basis of $\Lambda_1 = T_p(M)^*$.

(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \ldots \wedge \theta^{i_r}\}_{1 \le i_1 < i_2 < \ldots < i_r \le n}$

Exercise: show this

is a basis of $\Lambda_r(p)$ for $r > 1$.

- **Therefore**:

$$\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\Rightarrow$ no diff. forms of degree $r > n$!

Example: For $p \in M = \mathbb{R}^3$ we have bases:

$$\Lambda_1 = \mathrm{span}(dx^1, dx^2, dx^3)$$
$$\Lambda_2 = \mathrm{span}(dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3)$$
$$\Lambda_3 = \mathrm{span}(dx^1 \wedge dx^2 \wedge dx^3)$$

Definition:

$(\dim(\Lambda) = 2^n)$

$\Lambda(p) := \bigoplus_{i=0}^{n} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted ω.

- These fields form the Grassmann algebra of differential forms, $\Lambda(M)$.

Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e., to consider linear maps that obey the Leibniz rule.

(similar to how we defined tangent vectors as derivations of $\mathcal{F}(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

Definition:

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called a derivation of degree k, if:

$$\Phi : \Lambda_s(M) \to \Lambda_{s+k}(M) \quad \text{for all } s$$

$$\Phi : \alpha \wedge \beta \to \Phi(\alpha) \wedge \beta + \alpha \wedge \Phi(\beta)$$

Leibniz rule

$$\text{for all } \alpha, \beta \in \Lambda(M).$$

Also:

Definition:

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called an anti-derivation of degree k, if for all $\alpha \in \Lambda^q(M)$, $\beta \in \Lambda(M)$:

$\Phi: \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all s and

$$\Phi: \quad \alpha \wedge \beta \to \Phi(\alpha) \wedge \beta + (-1)^q \alpha \wedge \Phi(\beta)$$

"Anti-Leibniz rule"

Proposition: (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-) derivation

$$\Phi: \Lambda(M) \to \Lambda(M)$$

is already fully determined by its action only on $\Lambda_0(M)$ and on a basis of $\Lambda_1(M)$.

The exterior derivative

The exterior derivative,

$$\boxed{d: \Lambda(M) \to \Lambda(M)}$$

is the anti-derivation of degree $K=1$ which is defined through:

a) $d: \Lambda_0(M) \to \Lambda_1(M)$
$d: \quad f \to df$ } action of d on $\Lambda_0(M)$

b) $d: dx^i \to 0$ for all i. } action of d on a basis of $\Lambda_1(M)$!

In a chart:

We had: $d: f(x) \to df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$

Now we have more generally:

$$\beta = \sum_{i_1<\ldots<i_s} \underbrace{\beta_{i_1,\ldots,i_s}(x)}_{\Lambda_0(M)} \wedge \underbrace{dx^{i_1} \wedge \ldots \wedge dx^{i_s}}_{\Lambda_s(M)} \in \Lambda_s(M)$$

recall: $f \wedge \omega = f\omega$ when $f \in \Lambda_0$ and $\omega \in \Lambda$

Q: So how to carry out $d: \beta \to d\beta$?

A: By applying the anti-Leibnitz rule:

$$d\beta = \sum_{i_1<\ldots<i_s, j} \frac{\partial \beta_{i_1,\ldots,i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$$

$$+ \underbrace{\text{terms of the form } d(dx^{i_1}\wedge\ldots\wedge dx^{i_s})}_{}$$

$= 0$ because when applying Leibniz rule to $d(dx^{i_1}\wedge\ldots\wedge dx^{i_s})$ we eventually arrive at $d(dx^{i}) = 0\ldots$!

Proposition: $d: \Lambda(M) \to \Lambda(M)$ obeys:

$$\boxed{d \circ d = 0}$$

Proof:

$$d\circ d(\beta) = \underbrace{\sum_{\substack{i_1<\ldots<i_s\\ j,k}} \underbrace{\frac{\partial^2 \beta_{i_1\ldots i_s}(x)}{\partial x^j \partial x^k}}_{\text{sym. in } j,k} \underbrace{dx^k\wedge dx^j}_{\substack{dx^k\wedge dx^j = -dx^j\wedge dx^k\\ \text{i.e. antisym. in } j,k}}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_s}}_{\Rightarrow \sum = 0}$$

$$= 0$$

Example:

- For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have:

e.g.: electric potential

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\, dx^i$$

- **Notice:** $\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right)$ is the "Gradient field ∇f of f"

- **Example:** Electric field $E = d\phi$ from potential ϕ.

Now assume $\gamma \in \Lambda_1(M)$ is an arbitrary (i.e. not necessarily gradient) covariant vector field:

$$\gamma = \sum_{i=1}^{3} \gamma_i(x)\, dx^i \quad \in \Lambda_1(\mathbb{R}^3)$$

Then:

$$\beta := d\gamma = \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j}\, dx^j \wedge dx^i$$

from $dx^j \wedge dx^i = - dx^i \wedge dx^j$

$i=j$ does not occur because $dx^i \wedge dx^i = 0$

$$= \sum_{i<j} \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

$$= - \overbrace{\left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1} \right)}^{\beta_3 :=} dx^1 \wedge dx^2$$

$$- \overbrace{\left(\frac{\partial \gamma_1}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^1} \right)}^{\beta_2 :=} dx^1 \wedge dx^3$$

$$- \underbrace{\left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2} \right)}_{\beta_1 :=} dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$$\beta = \nabla \times \gamma$$

It is called a "pseudo vector field"

It is really a 2-form field in 3 dim.

Recall:

Gradient vector fields are curl free: $\nabla \times (\nabla f) = 0$

This is a special case of

$$d \circ d = 0$$

because if $\beta = d\gamma$ then:

$$d\beta = d^2\gamma = 0$$

Definition:

A differential form ω is called **closed** if:

$$d\omega = 0$$

E.g.: We saw that $\beta := d\gamma$ is closed. Is this example typical?

A differential form ω is called **exact** if there exists a ν so that

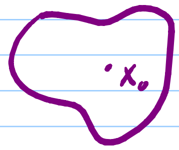
$$\omega = d\nu$$

(ν is like an anti-derivative!)

How are closedness and exactness related?

This actually depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

Simplest case: Assume $\mathcal{M}$ is contractible



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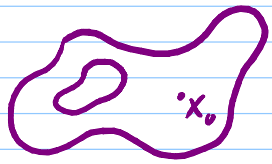
i.e., $\exists\, \mathcal{F}: [0,1] \times \mathcal{M} \to \mathcal{M}$ (continuous)

so that $\mathcal{F}(0,x) = x \quad \forall x$

$\mathcal{F}(1,x) = x_0 \quad \forall x$

(some fixed pt $\in \mathcal{M}$)



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Poincaré lemma:

On any contractible manifold:

$$\gamma \text{ exact} \Longleftrightarrow \gamma \text{ closed}$$

E.g.

- $\mathbb{R}^n$ is contractible
- $\mathbb{R}^n \setminus \{p\}$ is not contractible (some arbitrary point $\in \mathcal{M}$)

In general: We only have

$$\gamma \text{ exact} \Rightarrow \gamma \text{ closed}$$

which is because $d^2 = 0$.

$\rightsquigarrow$ We obtain a tool for classifying the "global topology" of differentiable manifolds (checking for holes, handles etc.)

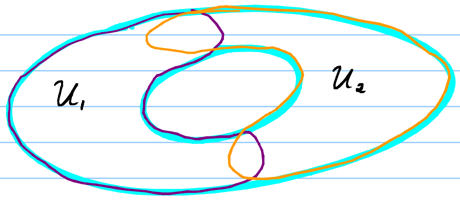
Def: This is called the "de Rham cohomology theory"

There are also other cohomology theories!

(which is good because none yields a complete classification)

Example: K-theory

- This cohomology theory uses the fact that:
- There is only one way to put a vector bundle on a contractible mfld but others can have many non-isomorphic vector bundles.
- Recall that, e.g., for a suitable vectorbundle B:



$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$$

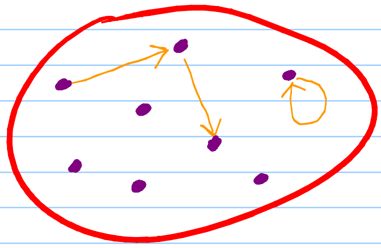
$$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$$

But: $B \not\simeq M \times \mathbb{R}^n$

Comment: All cohomology theories are:

"Natural Transformations" between two "Categories".

Def: A category is a set of objects and morphisms:



Axioms:

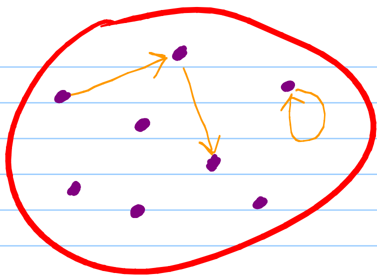
□ $\exists\ A \to B$ and $B \to C \Rightarrow \exists\ A \to C$

□ Associativity:

$(\bullet \to \bullet \to) \bullet \to \; = \; \bullet \to (\bullet \to \bullet \to)$

□ $\forall \bullet \ \exists \ \bullet \circlearrowleft$

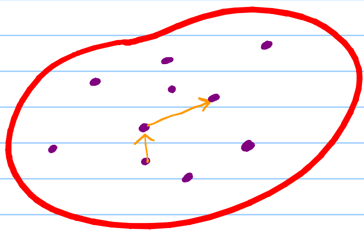
Examples:



Category of vector spaces **Vec**:

Objects: all vector spaces

Morphisms: linear transformations

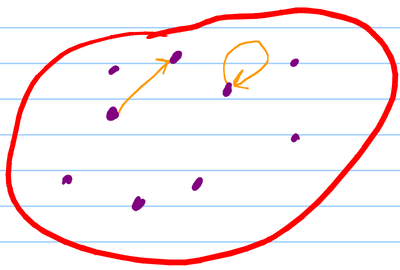


Category of associative algebras **Alg**:

Objects: all assoc. algebras

Morphisms: algebra homomorphisms

But also:

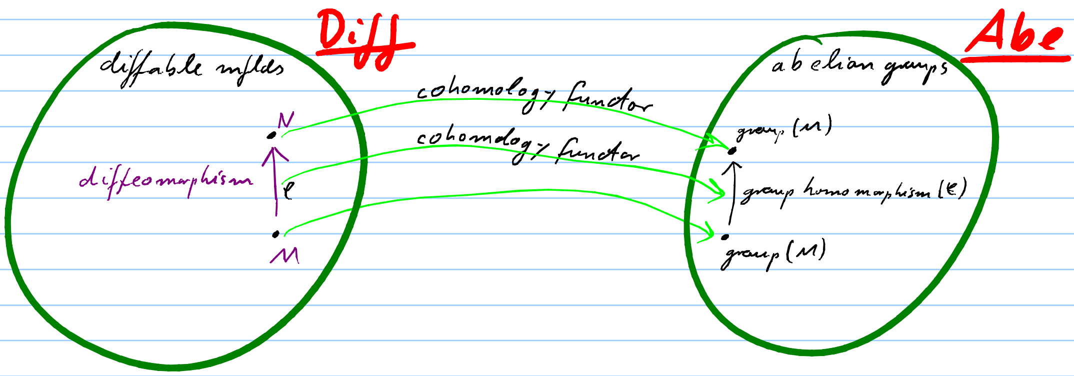


Category of categories **Cat**:

Objects: all categories

Morphisms: natural transformations also called functors.

A cohomology theory is a morphism between two objects in **Cat**, namely **Diff** and **Abe**:



Crucial: group(M) not homomorph group(N) $\Rightarrow$ M not diffeomorph to N

Note: This is what category theory was originally developed for.

In this course:

We'll focus on the local properties of the manifolds, such as curvature.

$\rightsquigarrow$ We mention cohomology theory issues only on the side in this course.

Next: The inner derivation of $\Lambda(M)$.