Differential forms (also called "exterior differential forms")

Why? Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p(M)^*$ at $p$:

\begin{itemize}
  \item Each $\omega \in T_p(M)^*$ is a lin. map:
    \[ \omega : T_p(M) \to \mathbb{R} \]
  \[ \omega : \xi \to \omega(\xi) \]
  \item Each such $\omega$ is a covariant tensor, of rank $(0,1)$
\end{itemize}

More generally, consider the covariant tensors of rank $(0,r)$:

\begin{itemize}
  \item Recall: $T_p(M)_r := \bigotimes^{r}_{\xi} T_p(M)^*$
  \item Each $\nu \in T_p(M)_r$ is a multi-linear map:
    \[ \nu : T_p(M)^* \to \mathbb{R} \]
  \item In particular, if $\xi_1, \ldots, \xi_r \in T_p(M)$ then:
    \[ \nu : \xi_1 \times \cdots \times \xi_r \to \nu(\xi_1, \ldots, \xi_r) \]
\end{itemize}
Definition: If $r \geq 1$ and $\omega \in T_p(M)_r$, then we define the "anti-symmetric part of $\omega" as the image $A(\omega)$ of $\omega$ under the linear antisymmetrization map $A$:

$$A(\omega)(\xi_1, \ldots, \xi_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) \omega(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(r)})$$

Why consider these? They will be key for integration! Only anti-symmetric cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

Concretely:

□ Consider $\omega := df \otimes dg$, for $f, g \in \mathcal{F}_p(M)$

□ Then $\omega(\xi, \xi_0) = df(\xi) \cdot dg(\xi_0)$ (which $\omega = \xi_0(f) \cdot \xi_0(g)$)

□ Apply $A$:

$$A(\omega)(\xi_1, \ldots, \xi_r) = \frac{1}{r!} \left( df(\xi_1) \cdot dg(\xi_r) - df(\xi_r) \cdot dg(\xi_1) \right)$$

□ $\Rightarrow$ We can also write:

$$A(df \otimes dg) = \frac{1}{r!} (df \otimes dg - dg \otimes df)$$
Proposition: A is a projector, i.e., it obeys
\[ A \circ A = A \]

Check in above example:
\[ A \circ A(d\phi \otimes dg) = A\left(\frac{1}{2}d\phi \otimes dg - \frac{1}{2}dg \otimes d\phi\right) \]
\[ = \frac{1}{2}\left(\frac{1}{2}d\phi \otimes dg - \frac{1}{2}dg \otimes d\phi\right) - \frac{1}{2}\left(\frac{1}{2}dg \otimes d\phi - \frac{1}{2}d\phi \otimes dg\right) \]
\[ = \frac{1}{2}(d\phi \otimes dg - dg \otimes d\phi) \]
\[ = A(d\phi \otimes dg) \]

Definition:
For \( r > 1 \) we define the space of differential \( r \)-forms (or 'exterior' \( r \)-forms) \( \Lambda^r(M) \) at \( p \in M \) as the subspace of totally antisymmetric tensors of rank \((0, r)\):
\[ \Lambda^r(p) := A T^r_p(M) \]

\[ \Rightarrow \text{So if } \nu \in \Lambda^r(p) \text{ then } \tilde{\nu} = A(\nu) = \nu \]
Definition: For $r=0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := IR$$ (for 0 forms on the entire manifold we will have $\Lambda_0 := \mathbb{R}$)

For $r=1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

$$\Lambda_1(p) := T_p(M),$$

Strategy now:
Define multiplication $\rightarrow$ obtain algebra $\rightarrow$ obtain derivations

The wedge product:

Definition: If $\omega \in \Lambda^r(p), \nu \in \Lambda^s(p)$, and $r, s \geq 0$ then the wedge product $\wedge$ yields a new differential form:

$$\wedge : \Lambda^r(p) \times \Lambda^s(p) \rightarrow \Lambda^{r+s}(p)$$

$$\wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \frac{(s+r)!}{s! \cdot r!} A(\omega \otimes \nu)$$

A normalization factor

Definition: For $c \in \mathbb{R}$, $\omega \in \Lambda^s$ we have $c \wedge \omega = c \omega$

Note: $dx^i \wedge dx^i = 0 \quad \forall i$

Example: For $dx^i, dx^i$ we obtain:

$$dx^i \wedge dx^i = (dx^i \otimes dx^i - dx^i \otimes dx^i)$$
Properties of $\wedge$

- **Bi-linear**: 
  \[(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta\]
  \[(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \text{ for } a \in \mathbb{R}\]

- **Associative**: 
  \[(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)\]

- **"Graded" commutative**: 
  \[\omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \text{ if } \omega, \nu \in \Lambda^r, \Lambda^s\]

  Example: \[dx^i \wedge e_i(p)\]
  
  \[dx^i dx^j = -dx^j dx^i \text{ since } i = j = 1\]

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- We can use $\wedge$ to build bases of $\Lambda^r(p)$:

  Assume: \[\{\Theta^i\}_{i=1}^m\] is a basis of $\Lambda^1 = T_p(n)^*$

  (for example $\Theta^i = dx^i$)

  Then: \[\{\Theta^{i_1} \wedge \Theta^{i_2} \wedge ... \wedge \Theta^{i_r}\}_{1 \leq i_1 < i_2 < ... < i_r \leq n}\]

  is a basis of $\Lambda^r(p)$ for $r > 1$.

  **Exercise**: show this

- Therefore:

  \[
  \dim(\Lambda^r(p)) = \binom{m}{r} = \frac{m!}{r!(m-r)!}
  \]

  \[\Rightarrow \text{ no diff. forms of degree } r > m\]
Example:
For $p \in M = \mathbb{R}^3$, we have bases:

- $\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$
- $\Lambda_2 = \text{span}(dx^1 dx^3, dx^1 dx^2, dx^2 dx^3)$
- $\Lambda_3 = \text{span}(dx^1 dx^2 dx^3)$

Definition:

$\Lambda(p) = \bigoplus_{i=0}^{n} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted $\omega$.

- These fields form the Grassmann algebra of differential forms, $\Lambda(M)$.
Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e., to consider linear maps that obey the Leibnitz rule.

(similar to how we defined tangent vectors as derivations of $F(M)$)

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

Definition:

A linear map $\Phi: \Lambda(M) \to \Lambda(M)$ is called a derivation of degree $k$, if:

$\Phi: \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all $s$

$\Phi: d \wedge \beta \to \Phi(d) \wedge \beta + d \wedge \Phi(\beta)$

for all $d, \beta \in \Lambda(M)$.

Also:
**Definition:**

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called an anti-derivation of degree $K$, if for all $\omega \in \Lambda^1(M), \beta \in \Lambda(M)$:

$\Phi : \Lambda_s(M) \to \Lambda_{s+K}(M)$ for all $s$ and

$\Phi : \omega \wedge \beta \to \Phi(\omega) \wedge \beta + (-1)^{sK} \omega \wedge \Phi(\beta)$

"Anti-Leibniz rule"

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**Proposition:** (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-)derivation

$\Phi : \Lambda(M) \to \Lambda(M)$

is already fully determined by its action only on $\Lambda_0(M)$ and on a basis of $\Lambda_1(M)$. 
The exterior derivative

The exterior derivative,

\[ d: \Lambda^k(M) \to \Lambda^{k+1}(M) \]

is the anti-derivation of degree \( k = 1 \) which is defined through:

1. \( d: \Lambda^0(M) \to \Lambda^1(M) \)
   \[ d: f \to df \]
2. \( d: dx_i \to 0 \) for all \( i \) in a basis of \( \Lambda^1(M) \)

In a chart:

We had:

\[ d: f(x) \to df(x) = \sum \frac{\partial f}{\partial x_i} dx_i \]

Now we have more generally:

\[ \beta = \sum_{i_1, \ldots, i_s} \beta_{i_1, \ldots, i_s}(x) dx_1 \wedge \cdots \wedge dx_s \in \Lambda^s(M) \]

Recall:

\[ f \wedge w = \sum_{i \leq j} f_i w_j \] when \( f \in \Lambda^0 \) and \( w \in \Lambda^k \)

Q: So how to carry out \( d: \beta \to d\beta \)?

A: By applying the anti-Leibniz rule:

\[ d\beta = \sum_{i_1, \ldots, i_s} \frac{\partial \beta_{i_1, \ldots, i_s}(x)}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge dx_s \]
Proposition: \( \Lambda(\mathbb{M}) \to \Lambda(\mathbb{M}) \) obeys:
\[
d \circ d = 0
\]

Proof:
\[
d d(\theta) = \sum_{\text{i, j, k}} \left( \frac{\partial^2 \theta}{\partial x^i \partial x^j}(x) \right) dx^i dx^j dx^k
\]

\[
= 0
\]

\( \Rightarrow \sum_i = 0 \)

Example:
- For \( M = \mathbb{R}^3 \) and \( f \in \mathbb{F}(M) \) we have:
\[
d f = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} dx^i
\]

Notice: \( \left( \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3} \right) \) is the "gradient field of \( f \)"

Example: Electric field \( E = d \phi \) from potential \( \phi \).
Now assume \( \gamma \in \Omega, (M) \) is an arbitrary (i.e. not necessarily gradient) covariant vector field:

\[
\gamma = \sum_{i=1}^{3} \gamma_i(x) \, dx^i \in \Omega, (\mathbb{R}^3)
\]

Then:

\[
\beta_i = d\gamma = \sum_{i,j} \frac{\partial \gamma_j(x)}{\partial x^i} \, dx^i \wedge dx^j
\]

\( i \neq j \) does not occur because \( dx^i \wedge dx^i = 0 \)

\[
\beta_i = \sum_{i \neq j} \left( \frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) \, dx^i \wedge dx^j
\]

Notice: \( (\beta_1(x), \beta_2(x), \beta_3(x)) \) are the components of the curl \( \beta = \nabla \times \gamma \)

It is called a "pseudo vector field"

It is really a 2-form field in 3 dim.
Recall:

Gradient vector fields are curl free: \( \nabla \times (\nabla f) = 0 \)

This is a special case of

\[ \text{curl } \mathbf{d} = 0 \]

because if \( \beta = dy \) then:

\[ d\beta = d^2 y = 0 \]

Definition:

A differential form \( \omega \) is called \text{closed} if:

\[ d\omega = 0 \]

E.g.: We saw that \( \beta := dy \) is closed. Is this example typical?

A differential form \( \omega \) is called \text{exact} if there exists a \( \nu \) so that

\[ \omega = d\nu \]

(\( \nu \) is like an anti-derivative)
How are closedness and exactness related?

This actually depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

Simpler case: Assume $M$ is contractible.

- $\exists \tilde{\gamma}$ continuous, i.e., $\exists \tilde{\gamma} : [0,1] \times M \rightarrow M$
- So that $\tilde{\gamma}(0,x) = x \quad \forall x$
- $\tilde{\gamma}(1,x) = x_0 \quad \forall x$
  $\uparrow$ some fixed $p \in M$

Poincaré lemma:

On any contractible manifold:

$\gamma$ exact $\iff$ $\gamma$ closed

E.g.

- $\mathbb{R}^n$ is contractible
- $\mathbb{R}^n \setminus \{p\}$ is not contractible
  $\uparrow$ some arbitrary point $p \in M$
In general: We only have
\[ \gamma \text{ exact } \Rightarrow \gamma \text{ closed} \]
which is because \( d^2 = 0 \).

We obtain a tool for classifying the "global topology" of
differentiable manifolds (checking for holes, bundles etc.).

Def: This is called the "de Rham cohomology theory."

There are also other cohomology theories!
(Which is good because none yields a complete classification)

Example: \( K \)-theory

- This cohomology theory uses the fact that:
- There is only one way to put a vector bundle on a contractible manifold but others can have many non-isomorphic vector bundles.
- Recall that, e.g., for a suitable vector bundle \( B \):
  \[ \pi^{-1}(U_1) \cong U_1 \times \mathbb{R}^m \]
  \[ \pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^m \]
  But: \( B \not\cong M \times \mathbb{R}^m \)
Comment: All cohomology theories are: "Natural Transformations" between two "Categories."

Def: A category is a set of objects and morphisms:

Axioms:

1. If \( A \rightarrow B \) and \( B \rightarrow C \), then \( A \rightarrow C \)

2. Associativity:

\[
(\bullet \rightarrow \bullet \rightarrow \bullet) = (\bullet \rightarrow \bullet) \rightarrow (\bullet) = \bullet \rightarrow (\bullet \rightarrow \bullet)
\]

Examples:

- Category of vector spaces \( \mathbf{Vec} \):
  - Objects: all vector spaces
  - Morphisms: linear transformations

- Category of associative algebras \( \mathbf{Alg} \):
  - Objects: all assoc. algebras
  - Morphisms: algebra homomorphisms

But also:

- Category of categories \( \mathbf{Cat} \):
  - Objects: all categories
  - Morphisms: natural transformations, also called functors.
A cohomology theory is a morphism between two objects in \textit{Cat}, namely \textbf{Diff} and \textbf{Abe}:

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {\text{Diff}};
  \node (B) at (4,0) {\text{Abe}};
  \node (M) at (0,-2) {M};
  \node (N) at (4,-2) {N};
  \draw[->] (A) -- (M);
  \draw[->] (A) -- (N);
  \draw[->] (B) -- (N);
  \draw[->] (B) -- (M);
  \node at (2,-1) {\text{diffeomorphism}};
  \node at (-1,-1) {\text{diffeomorphism}};
  \node at (2,-3) {\text{abelian groups}};
  \node at (-1,-3) {\text{cohomology functor}};
  \node at (1,-3) {\text{group homomorphism}};
  \node at (3,-3) {\text{group homomorphism}};
  \node at (1,-4) {\text{group (M)}};
  \node at (3,-4) {\text{group (N)}};
\end{tikzpicture}
\end{center}

\textbf{Crucial:} \text{group}(M) \not\cong \text{group}(N) \Rightarrow M \not\cong N

\textbf{Note:} This is what category theory was originally developed for.

\textbf{In this course:}

We'll focus on the local properties of the manifolds, such as curvature.

\implies We mention cohomology theory issues only on the side in this course.

\textbf{Next:} The inner derivation of $\Lambda(M)$. 