

Recall: \square The set $\Lambda(M)$ of differential forms on M is an associative algebra, called the Grassmann algebra over M .

\square The multiplication in $\Lambda(M)$ is the wedge product: $\wedge: \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$

\square The exterior derivative $d: \Lambda(M) \rightarrow \Lambda(M)$ is an anti-derivation of degree $k=1$ of the Grassmann algebra $\Lambda(M)$.

But: How to obtain a directional derivative on $\Lambda(M)$?

Recall: Tangent vectors ξ are directional derivatives on $\Lambda_0(M)$!

Plan now:

A. Define an anti-derivation i_ξ of degree $k=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p -form.)

B. Combine d, i_ξ to obtain a derivation of degree $k=0$: the Lie derivative

(And the Lie derivative is going to be the directional derivative for differential forms and tensors)

A. The "Inner Derivation":

(Idea: A tangent vector, ξ , maps
 $\xi: \Lambda_1 \rightarrow \Lambda_0$
 $\xi: \omega \rightarrow \omega(\xi)$
This can be extended to all of Λ .)

- Assume ξ is a tangent vector field.
- Our aim: to define an anti-derivation, i_ξ , of degree $k = -1$, i.e., a linear map

$$i_\xi: \Lambda_s(\mathcal{M}) \rightarrow \Lambda_{s-1}(\mathcal{M})$$

$$i_\xi: \omega \rightarrow i_\xi(\omega)$$

which obeys the anti-Leibniz rule:

$$i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^r \omega \wedge i_\xi(\nu)$$

if $\omega \in \Lambda_r(\mathcal{M})$.

- Definition:

$$i_\xi: \Lambda_0 \rightarrow 0$$

$$i_\xi: \Lambda_1 \rightarrow \Lambda_0$$

$$i_\xi: \omega \rightarrow \omega(\xi)$$

- Recall: By linearity and the anti-Leibniz rule this already defines $i_\xi: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$.

- Proposition: If $\gamma \in \Lambda_s(\mathcal{M})$ then $i_\xi(\gamma) \in \Lambda_{s-1}(\mathcal{M})$ maps $(s-1)$ tangent vectors $\eta_1, \dots, \eta_{s-1}$ this way:

$$i_\xi(\gamma)(\eta_1, \eta_2, \dots, \eta_{s-1}) := \gamma(\xi, \eta_1, \eta_2, \dots, \eta_{s-1})$$

Example: * Consider $\gamma \in \Lambda_2(M)$ $\gamma := \omega \wedge v$ $\omega \in \Lambda_1(M)$ $v \in \Lambda_1(M)$

* What is $i_\xi(\gamma) \in \Lambda_1(M)$? Leibniz rule \Rightarrow

$$i_\xi(\gamma) = i_\xi(\omega \wedge v) = i_\xi(\omega) \wedge v + (-1)^1 \omega \wedge i_\xi(v) \\ = \omega(\xi) v - v(\xi) \omega$$

* Apply $i_\xi(\gamma) \in \Lambda_1(M)$ to a tangent vector η :

$$i_\xi(\gamma)(\eta) = \omega(\xi) v(\eta) - v(\xi) \omega(\eta)$$

* Compare with claim of proposition:

$$i_\xi(\gamma)(\eta) = i_\xi(\omega \wedge v)(\eta) = i_\xi(\omega \otimes v - v \otimes \omega)(\eta) \\ = \omega(\xi) v(\eta) - v(\xi) \omega(\eta) \quad \checkmark$$

Recall: $\omega \wedge v = \omega \otimes v - v \otimes \omega$

Properties of i_ξ :

$$\square \quad i_{\xi_1} \circ i_{\xi_2} = -i_{\xi_2} \circ i_{\xi_1}$$

Thus, in particular:

$$i_\xi \circ i_\xi = 0$$

(Exercise: prove this)

Recall: We also have $d \circ d = 0$

(Simply the evaluation of a dual vector applied to a vector in the vector space.)

Recall: For $\xi \in T_p(M)$, $\gamma \in T_p^*(M)$, we have $i_\xi(\gamma) = \gamma(\xi) = \xi(\gamma)$

Definition: The inner derivation, $i_\xi(\gamma)$, of a $\gamma \in \Lambda(M)$ is also called the interior product of ξ and γ .

B. The Lie derivative, L_ξ : (algebraic definition)

Vectors $\xi: \Lambda_0(M) \rightarrow \Lambda_0(M)$ are directional derivatives.

How to generalize the notion of directional derivative to all of $\Lambda(M)$?

We have: \square $d: \Lambda_s(M) \rightarrow \Lambda_{s+1}(M)$ generalizes the notion of differential $d: \Lambda_0 \rightarrow \Lambda_1$, $d: f \rightarrow df$ to all of $\Lambda(M)$.

\square $i_\xi: \Lambda_s(M) \rightarrow \Lambda_{s-1}(M)$ generalizes the notion of evaluation of vectors ξ on covectors $\omega \in \Lambda_1(M)$ to all of $\Lambda(M)$.

Spoiler: It will be: $L_\xi = d \circ i_\xi + i_\xi \circ d$

To construct L_ξ , let us first collect desired properties:

\square As a directional derivative, it should be a derivation, not an anti-derivation, i.e.:

$$L_\xi(\omega \wedge \nu) = L_\xi(\omega) \wedge \nu + \omega \wedge L_\xi(\nu)$$

(Recall that the directional derivatives on functions $\Lambda_0(M)$, namely the tangent vectors, are mapping $\Lambda_0(M) \rightarrow \Lambda_0(M)$)

\square L_ξ should map r -forms into r -forms:

$$L_\xi: \Lambda_r(M) \rightarrow \Lambda_r(M)$$

i.e. it should be of degree $K=0$. In particular:

- On functions $f \in \mathcal{F}(M) = \Lambda_0(M)$ it should be the usual directional derivative:

$$L_\xi : \Lambda_0(M) \rightarrow \Lambda_0(M)$$

$$L_\xi : f \rightarrow \xi(f) \quad \left(= \sum_{i=1}^n \xi^i(x) \frac{\partial}{\partial x^i} f(x) \right)$$

- Recall: once we define L_ξ on Λ_0 and a basis of $\Lambda_1(M)$, then by linearity and the Leibniz rule, L_ξ will automatically be defined on all of $\Lambda(M)$.
- Consider, therefore, any $df \in \Lambda_1(M)$, e.g., the basis vectors $df = dx^i$.
recall that df is the gradient vector field of the function f .

- Then it is natural to define the directional derivative of a gradient field of a function to be the gradient of the directional derivative of the function:
(because derivatives ought to commute and the gradient is a derivative too.)

$$L_\xi : \Lambda_1(M) \rightarrow \Lambda_1(M)$$

$$L_\xi : df \rightarrow d(\xi(f))$$

$\underbrace{\hspace{10em}}_{\in \Lambda_0(M)}$
 $\underbrace{\hspace{10em}}_{\in \Lambda_1(M)}$

i.e.: $L_\xi(df) = d(\xi(f))$ (D)

directional derivative of gradient = gradient of directional derivative

Question: Now that L_{ξ} is a fully defined derivation

$$L_{\xi}: \Lambda(M) \rightarrow \Lambda(M),$$

can we relate it to d and i_{ξ} ? **Yes:**

Exercise: show it is a derivation

Cartan's equation:

$$L_{\xi} = d \circ i_{\xi} + i_{\xi} \circ d$$

Proof:

check on $\Lambda_0(M)$: $L_{\xi} f = d \circ i_{\xi}(f) + i_{\xi}(df) = 0 + d f(\xi) = \xi(f)$ ✓

$\Lambda_0(M)$
↓
 $f \in \Lambda_0(M)$
 $= 0$ because
 $= df(\xi) = \xi(f)$
because: $d^2 = 0$

check on basis of $\Lambda_1(M)$, e.g. $df = dx^i$: $L_{\xi} df = d \circ i_{\xi}(df) + i_{\xi} \circ ddf = d(\xi(f))$ ✓

J.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative

Definition:

For any linear maps $A: \Lambda(M) \rightarrow \Lambda(M)$, $B: \Lambda(M) \rightarrow \Lambda(M)$

we define their commutator (or Lie-, or Poisson bracket):

$$[A, B] := A \circ B - B \circ A$$

Examples of maps:

$$d: \Lambda(M) \rightarrow \Lambda(M)$$

$$i_{\xi}: \Lambda(M) \rightarrow \Lambda(M)$$

$$L_{\xi}: \Lambda(M) \rightarrow \Lambda(M)$$

For the commutators of d , i_{ξ} and L_{ξ} one can prove:

Proposition:

$$\square [L_{\xi}, d] = 0$$

$$\square [L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1, \xi_2]}$$

$$\square [L_{\xi_1}, \iota_{\xi_2}] = \iota_{[\xi_1, \xi_2]}$$

Exercise: prove this

Here we used on the right hand side that also vector fields

$$\xi: \Lambda_0(M) \rightarrow \Lambda_0(M),$$

have commutators:

$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1}^m (\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}) \frac{\partial}{\partial x^j} f$$

$$= \sum_{i,j=1}^m (\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}) \frac{\partial}{\partial x^j} f$$

$$= \sum_{j=1}^m v^j \frac{\partial}{\partial x^j} f = v(f)$$

The terms with the second derivatives cancel because:
 $\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f$

Questions:

Since L_{ξ} is the directional derivative on $\Lambda(M)$:

■ Can L_{ξ} be extended to a directional derivative for all tensor fields? **Yes!**

■ Can L_{ξ} be expressed as a Newton-Leibniz limit similar to

need an analog: a shift on a manifold, in the direction given by ξ .

$$f'(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

? **Yes!**

To this end:

The geometric definition of L_{ξ} :

□ Recall that for any path

$$\begin{aligned} \gamma: \mathbb{R} \supset J &\rightarrow M && \text{an open interval of } \mathbb{R} \\ \gamma: t &\rightarrow \gamma(t) \end{aligned}$$

we have a tangent vector $\bar{\gamma}(t) \in T_{\gamma(t)}(M)$ at each point $\gamma(t)$ of the path:

$$\bar{\gamma}(t): f \rightarrow \bar{\gamma}(t)(f) = \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=t_0}$$

(the geom. definition of the tangent space)

□ Definition: For a given vector field, ξ , a path γ is called an **integral curve** of ξ , if

$$\bar{\gamma}(t) = \xi(\gamma(t))$$

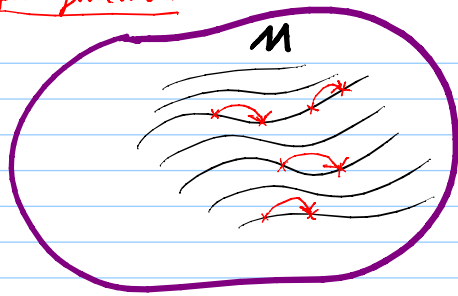
↑ path's velocity vector at $\gamma(t)$ ↑ vector of field ξ at $\gamma(t) \in M$.

□ From theory of ODEs:

For every $p \in M$ there exists a maximal (i.e. inextendible) C^∞ integral curve through p .

□ Thus, ξ yields a "flow": (at least for small t , locally):

for a fixed t :



i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M .

□ The flow is a diffeomorphism " $\phi_t: M \rightarrow M$ ":



□ As always, a diffeomorphism of manifolds induces

corresponding isomorphisms of the tangent, cotangent and all tensor spaces at p and at $\phi_t(p)$ respectively:

$$\phi_t^*: T_p(M)_s^r \rightarrow T_{\phi_t(p)}(M)_s^r$$

□ Recall: A tensor field τ assigns to each $p \in M$ a tensor $\tau(p) \in T_p(M)_s^r$.

Definition:

We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\phi_t^*(\tau(p)) = \tau(\phi_t(p)) \quad \forall t \forall p$$

(The flow produces an image of M in M .)

image of the tensor field's value at p

tensor field's value at the image of p

□ Definition:

The **Lic derivative** of any tensor field τ at the point $p = \gamma(0) \in \mathcal{M}$ with respect to the flow induced by a vector field ξ is defined through:

$$L_{\xi} \tau := \lim_{t \rightarrow 0} \frac{1}{t} \left(\phi_t^* \tau - \tau \right)$$

geom. definition

Tensor field value at image of p , i.e. $\in T_{\gamma(t)} \mathcal{M}$

$$\text{i.e. } L_{\xi}(\tau)(p) = \lim_{t \rightarrow 0} \frac{1}{t} \left[\underbrace{(\phi_t^*)^{-1}(\tau(\gamma(t)))}_{\in T_p(\mathcal{M})} - \underbrace{\tau(p)}_{\uparrow = \gamma(0)} \right]$$

Explicitly, in a chart:

□ $\phi: x \rightarrow \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t \xi^i(x) + \mathcal{O}(t^2)$

□ Jacobian matrix: $\frac{\partial \tilde{x}^i}{\partial x^j} = \delta_j^i + t \frac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$

we write $= \xi_{,j}^i$

□ Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta_j^i - t \frac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$

□ Image of tensor at $\tau(\tilde{x})_{j_1 \dots j_r}^{i_1 \dots i_r}$ under flow, backwards, $\tilde{x} \rightarrow x$, has the

From now, we will omit writing Σ : Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\begin{aligned} \phi_t^* (\tau(x))_{j_1 \dots j_r}^{i_1 \dots i_r} &= \tau_{j_1 \dots j_r}^{i_1 \dots i_r}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{i_r}}{\partial \tilde{x}^{j_r}} \frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}} \dots \frac{\partial \tilde{x}^{j_r}}{\partial x^{i_r}} \\ &= \tau_{j_1 \dots j_r}^{i_1 \dots i_r}(x + t\xi) \left(\delta_{j_1}^{i_1} - t \xi_{j_1}^{i_1} \right) \dots \left(\delta_{j_r}^{i_r} - t \xi_{j_r}^{i_r} \right) \\ &\quad \cdot \left(\delta_{j_1}^{i_1} + t \xi_{j_1}^{i_1} \right) \dots \left(\delta_{j_r}^{i_r} + t \xi_{j_r}^{i_r} \right) + \mathcal{O}(t^2) \end{aligned}$$

$$\begin{aligned}
 &= \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) + t \tau_{j_1 \dots j_s, \kappa}^{i_1 \dots i_s}(x) \xi^\kappa(x) \\
 &\quad - t \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, i_1}^{i_1}(x) - \dots - t \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, i_r}^{i_r}(x) \\
 &\quad + t \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, j_1}^{j_1}(x) + \dots + t \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, j_s}^{j_s}(x)
 \end{aligned}$$

$f_{, \kappa} := \frac{\partial}{\partial x^\kappa} f$

$$\begin{aligned}
 \Rightarrow (L_\xi \tau)_{j_1 \dots j_s}^{i_1 \dots i_s}(x) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\phi^{*t}(\tau(x))_{j_1 \dots j_s}^{i_1 \dots i_s} - \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x(0)) \right) \\
 &= \tau_{j_1 \dots j_s, \kappa}^{i_1 \dots i_s}(x) \xi^\kappa(x) - \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, i_1}^{i_1}(x) - \dots \\
 &\quad + \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, j_1}^{j_1}(x) + \dots + \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi_{, j_s}^{j_s}(x)
 \end{aligned}$$

□ Equivalent to algebraic definition of L_ξ ?

Yes: Check, e.g., that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

□ For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

$$L_\xi \tau(x) = \xi^\kappa \tau_{, \kappa} = \xi^\kappa \frac{\partial}{\partial x^\kappa} \tau(x) \text{ is gradient} \checkmark$$

□ Co-Vector field: $\tau = \tau_j(x) dx^j \in \Lambda_1(M)$

$$L_\xi \tau(x) = \left(\xi^\kappa(x) \tau_{j, \kappa}(x) + \tau_\kappa(x) \xi_{, j}^\kappa(x) \right) dx^j$$

Exercise: verify that this agrees with the algebraically defined action of L_ξ on $\Lambda_1(M)$.

▮ Collected properties: (without proof)

▮ $L_\xi : T_p(M)_s \rightarrow T_p(M)_s$ (i.e. not just $\Lambda_s \rightarrow \Lambda_s$)

▮ In particular, the Lie derivative of a vector field η is:

$$L_\xi : \eta \rightarrow L_\xi(\eta) = [\xi, \eta]$$

▮ One also finds:

$$L_{\xi+\eta} = L_\xi + L_\eta$$

$$L_{[\xi, \eta]} = [L_\xi, L_\eta] \quad (= L_\xi \circ L_\eta - L_\eta \circ L_\xi)$$

▮ Does it still obey a Leibniz rule?

Yes: $L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$

(tensors form an algebra w. respect to multiplication \otimes)