Recall: Physical motivation for the "Metric Tensor"

- In Minkowski space, in inertial and cartesian coordinates:
  \[
  \left[ \text{distance } (x, y) \right]^2 = -(x^0 - \hat{x}^0)^2 + (x^1 - \hat{x}^1)^2 + (x^2 - \hat{x}^2)^2 + (x^3 - \hat{x}^3)^2
  \]
  up to choice of inertial c.s.
  with \( \eta_{\mu \nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

- In Minkowski space, in an arbitrary coordinate system:
  \[
  \left[ \text{distance } (x, \tilde{y}) \right]^2 = g_{\mu \nu}(x)(x^\mu - \hat{x}^\mu)(\tilde{x}^\nu - \hat{\tilde{x}}^\nu) + O^3
  \]
  \( (\text{e.g., polar c.s., \( \omega \))} \) with \( g_{\mu \nu}(x) \neq \eta_{\mu \nu} \)

Generalization to curved space-time, historically:
Allow even such \( g_{\mu \nu}(x) \) which in no coordinate system obey:
\[
g_{\mu \nu}(x) = \eta_{\mu \nu} \quad \text{for all } x \in M
\]
\( \Rightarrow \) \( g_{\mu \nu}(x) \) is not simply \( g_{\mu \nu} \) in noninertial coordinates
\( \Rightarrow \) Such \( g_{\mu \nu}(x) \) take us beyond special relativity!

Enforce Einstein's equivalence principle:
Require \( g_{\mu \nu} \) to be such that
\[
(\text{Recall special principles (EP):})

\begin{align*}
\text{If freely falling small sources} & \quad \text{\Rightarrow \ "weak EP."} \\
\text{\( + \) some internal source physics \( \Rightarrow \) \"Causal EP.\")} & \quad \text{\( + \) some internal source physics \( \Rightarrow \) \"Strong EP.\")}
\end{align*}
\]
for each \( x \in M \) there exists a coordinate system so that at least at \( x \):
\[
g_{\mu \nu}(x) = \eta_{\mu \nu} \quad \text{(i.e., locally special relativity holds)}
\]
\( \text{to lowest meaningful order.} \)
Recall: Math. definition of the metric tensor:

\( g \) is covariant tensor of rank \((0,2)\)  
(because \( g \) is in special relativity)  
\( \text{e.g. } g^0_0 = dx^0 \)

Thus, if \( n \) cotangent vector fields \( \Theta^r(x) \)  
form bases at each point \( x \), then  
\( g \) is of the form:

\[ g(x) = g_{\mu\nu}(x) \Theta^\mu(x) \otimes \Theta^\nu(x) \]

Recall: \( g_{\mu\nu}(x) = g_{\nu\mu}(x) \) and \( g_{\mu\nu} \) is invertible (since nondegenerate)

\( g_{\mu\nu}(x) \) invertible \( \Rightarrow \) there exists a tensor \( \tilde{g} \) of rank \((2,0)\):

\[ \tilde{g}(x) = g^{\mu\nu}(x) \Theta_\mu(x) \otimes \Theta_\nu(x) \text{ with } g^{\mu\nu}(x) g_{\mu\nu}(x) = \delta^\rho_\sigma \]

Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point \( p \in M \) there is a coordinate system so that, when choosing the bases \( \{ dx^\mu \}, \{ \delta^\nu_\mu \} \)

then  
\( g(x) = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \),  
\( g_{\mu\nu}(x) = g(x) \delta^\nu_\mu \)

obeys:  
\( g_{\mu\nu}(p) = \eta_{\mu\nu} \) \( \text{ (in general only at } p \) \)

Modern formulation of the Einsteinian equivalence principle:

Independehtly of any choice of coordinate system:

There are choices of dual bases \( \{ \Theta^\mu(x) \}, \{ e_\nu(x) \} \) of \( T_x(M), T^*_x(M) \)

so that:

\[ g_{\mu\nu}(x) = g (e_\mu(x), e_\nu(x)) = \eta_{\mu\nu} \quad \forall x \in M \]
Now, knowing distances through $g_{\mu\nu}$, what else follows?

- Distances yield volumes, namely $g_{\nu}(x)$ induces an $\Omega(x)$.
- $g$, $g^*$ yield duality of covariance and contravariance.
- $g$ yields "Hodge star" $\star: \Lambda_p \rightarrow \Lambda_{n-p}$ duality.

$\star$ yields $(\omega, \omega)$ making the $\Lambda_p$ Hilbert spaces.

- $g$ yields co-derivative $\delta: \Lambda_p \rightarrow \Lambda_{p-1}$.
- $d, \delta$ yield the Laplacian/d'Alembertian $\Delta: \Lambda_p \rightarrow \Lambda_p$.

\[ \Rightarrow \] We can formulate wave equations on $M$!

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**Proposition:**

Given a notion of distance, i.e., a metric, $g$, this also induces a volume form $\Omega$. (i.e., a positive $\Omega \in \Lambda^n(M)$, i.e., that when integrated over any portion of $M$ yields a positive number)

Namely:

- Assume, as always, that $M$ is oriented.
- Consider a positive chart. (i.e., has positive $\det(Jacobian)$ with given atlas)

Then:

\[ \Omega := \sqrt{|\det(g_{\mu\nu}(x))|} \]

\[ \Omega := \sqrt{|\det(g_{\mu\nu}(x))|} \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \]

is a well-defined volume form.
Proof: a) Non-zero for all $p < m$?

Yes, because $g$ is assumed non-degenerate.

b) Well-defined, i.e., in definition chart-independent?

Yes: To see this, change chart: $x \rightarrow \tilde{x}$

Then: $\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial \tilde{x}^r}{\partial x^i} \frac{\partial \tilde{x}^s}{\partial x^j}$, because covariance i.e., as matrix:

$\tilde{g} = \left( \frac{\partial \tilde{x}^r}{\partial x^i} \right)^T \frac{\partial \tilde{x}^s}{\partial x^j} \frac{\partial x^i}{\partial \tilde{x}^r} \frac{\partial x^j}{\partial \tilde{x}^s}$, now take determinant:

$\Rightarrow |\tilde{g}| = \left| \frac{\partial \tilde{x}^r}{\partial x^i} \right|^2 |g|$, i.e., $|\tilde{g}|^{1/2} = \left| \frac{\partial x^i}{\partial \tilde{x}^r} \right| |g|^{1/2}$

Also: $d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n = det(\frac{\partial \tilde{x}^r}{\partial x^i}) dx^1 \wedge \ldots \wedge dx^n$

$\Rightarrow |\tilde{g}|^{1/2} d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n = \left[ \frac{\partial \tilde{x}^r}{\partial x^i} \right] \left[ \frac{\partial x^i}{\partial \tilde{x}^r} \right] |g|^{1/2} dx^1 \wedge \ldots \wedge dx^n$

Notation: ($\Omega$ is an $n$-form. What are its coefficients, as a covariant $(0,n)$ tensor?)

- Define:

$\varepsilon_{i_1 \ldots i_m} = \begin{cases} +1 & \text{if } (i_1, i_2, \ldots, i_m) \text{ is even permutation of } (1, 2, \ldots, n) \\ -1 & \text{if } (i_1, i_2, \ldots, i_m) \text{ is odd permutation of } (1, 2, \ldots, n) \\ 0 & \text{else} \end{cases}$

Unlike in SRT, $\varepsilon_{i_1 \ldots i_m}$ is not canonical, because $\Omega$ is:

- Then, $\Omega$ also needs:

$\Omega = \sqrt{|g|} \, d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n$ (n-form)

$\Rightarrow \Omega = \sqrt{|g|} \, \varepsilon_{i_1 \ldots i_m} \, dx^{i_1} \wedge \ldots \wedge dx^{i_m}$

$\Rightarrow \Omega = \Omega_{i_1 \ldots i_m} \, dx^{i_1} \wedge \ldots \wedge dx^{i_m}$ (covariant tensor)

$\Omega$ is called the "canonical" or "(pseudo)Riemannian", or "metric", volume form.
Q: Other use of \( g \)?

A: One needs \( g \) to formulate \( \Delta \), or \( \Box \), for wave equations.

Why? a) \( \Box \) should be non-directional 2nd derivative, but \( d^2 \) not.

b) need e.g. \( \delta A \to \Delta A \) found \( \Box \), i.e., need deep of forms conserved by \( \Box \).

**Strategy:**

A) Use \( g \) for a covariant \( \leftrightarrow \) contravariant tensors relation

B) Define a map "Hodge\( ^{\ast} \)" : \( \Lambda^r \to \Lambda^{n-r} \)

C) Define the "Codervivative": \( \delta \) : \( \Lambda^r \to \Lambda^{r-1} \)

D) Define "Laplacian/d'Alembertian" \( \Delta := \delta \delta + \delta d \)

Then, e.g., the Klein Gordon equation reads:

\[
(\Box + m^2) \phi = 0
\]

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A) **Covariant \( \leftrightarrow \) contravariant tensors equivalence through \( g \):**

- \( g(x) \) can be used as a map: by evaluation of one tensor factor:

\[
g(x) : \mathbf{T}_x^r(M) \to \mathbf{T}_{x}^r(M) \quad \text{by} \quad \Theta^i(x) \to \Theta^i(x) = g^i{}_{j} \Theta^j(x)
\]

\[
g(x) : \mathbf{\varepsilon}^i(x) \mathbf{e}_i(w) \to g_{mu}(x) \Theta^u(x) \Theta^v(x) \mathbf{\varepsilon}^v(x) \mathbf{e}_m(w)
\]

\[
e \in \mathbf{\varepsilon}^i(x) \mathbf{e}_i(w) \quad \in \mathbf{T}_x^r(M)
\]

\[
\Rightarrow \text{For the coefficient functions we have: } g : \mathbf{v}(x) \to \mathbf{w}_x(x) = g_{uv}(x) \mathbf{v}_x(x) \quad (\text{relative to bases } \Theta^i, e_j)
\]

Conversely, \( g^{-1} \) acts as:

\[
g^{-1}(x) : \mathbf{T}_x^r(M) \to \mathbf{T}_x^r(M)
\]

\[
g^{-1}(x) : \mathbf{w}_x(x) \to \mathbf{v}_x(x) = g^{uv}(x) \mathbf{w}_x(x)
\]

In this way, \( g, g^{-1} \) can lower or raise any tensor index, e.g.:

\[
g : t^{ij}_{\phantom{ij}k} \to t^{ij}_{\phantom{ij}k} = g^{is} t^{sj}_{\phantom{sj}k}
\]

and:

\[
g : t^{ij}_{\phantom{ij}k} \to t^{ij}_{\phantom{ij}k} = g_{ik} t^{ij}_{\phantom{ij}k}
\]
B) The Hodge $\star$ map: $\Lambda_p \rightarrow \Lambda_{n-p}$

Recall:
\[
\dim(\Lambda_p) = \binom{n}{p} = \frac{(n-p)!}{p!(n-p)!}
\]

Idea:
- Each $\omega \in \Lambda_p$ is a covariant tensor.
- Through $g$, it is equivalent to a contravariant tensor $\tilde{\omega}$.
- One can feed $\Omega$ with $\tilde{\omega}$ to obtain $\star \omega \in \Lambda_{n-p}$.

Concretely:
- Consider any $\omega = \frac{1}{p!} \omega^{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Lambda_p$.
- One could choose other bases.
- Consider any $\omega = \frac{1}{p!} \omega^{i_1, \ldots, i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}$.

- Use $g$ to define a contravariant image of $\omega$:

\[
\tilde{\omega} = \tilde{\omega}^{i_1, \ldots, i_p} = \frac{1}{n-p!} g_{j_1, 1} g_{j_2, 2} \cdots g_{j_{n-p}, n-p} \omega^{i_1, \ldots, i_p}
\]

where $\tilde{\omega}^{i_1, \ldots, i_p} = g_{j_1, i_1} g_{j_2, i_2} \cdots g_{j_{n-p}, i_{n-p}} \omega^{i_1, \ldots, i_p}$.

- Apply $\Omega$ on $\tilde{\omega}$:

\[
\Omega(\tilde{\omega}) = \Omega_{i_1, \ldots, i_{n-p}} \tilde{\omega}^{i_1, \ldots, i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_{n-p}} \in \Lambda_{n-p}
\]

- Define $\star \omega := \Omega(\tilde{\omega})$, i.e.:

\[
\star \omega = \frac{1}{(n-p)!} \omega^{i_1, \ldots, i_{n-p}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-p}}
\]

**Proposition:**

Assume $\omega \in \Lambda_p$. Then $\star(\star \omega) = (-1)^{p(n-p)+s} \omega$.

E.g., $s=1$ for space-time.

What is $s$? The "signature" of $g$ is $\text{sgn}(s) = (r, s)$, where in diagonal form: $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. 
Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

**Definition:** The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

$$ (\omega, \beta) := \sum_{p=0}^{d} \int_{M} \omega \wedge \beta $$

This definition is extended linearly to forms that are lin. comb. of forms of even degree, $p$.

**Notes:**
- If $g$ is indefinite, then also $(,)$ is indefinite.
- If $g$ is positive definite, i.e., if $\mathcal{M}$ is Riemannian, then $(,)$ is positive definite and $\Lambda$ becomes a Hilbert space.

#### 3) $(,)$ yields an adjoint for $d$, the Co-derivative $\delta$:

**Recall:** For any operator $A : D_A \subset X \to X$ (with $D_A$ dense, i.e., $\overline{D_A} = X$), its adjoint $A^+$ is defined to have the domain

$$ D_A^+ := \{ v \in X \mid \exists w \in X \forall z \in D_A : \langle v, Az \rangle = \langle w, z \rangle \} $$

and this action: $A^+v := w$. We then have:

$$ \langle A^+v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_A^+ $$

**Definition:**

The co-derivative, $\delta$, is the (anti-)adjoint of $d$ with respect to the inner product $(,)$ on $\Lambda(M)$:

$$ (\delta \omega, \beta) := - (\omega, d\beta) \quad \forall \omega \in \delta \omega, \beta \in D_\omega $$
C) The Codifferential $\delta$ Explicitly

Clearly: $\delta : \Lambda^p(M) \to \Lambda^{p+1}(M)$

Proposition: $\delta : \omega \mapsto \left( (-1)^{p+q+r+s} \ast d \ast \delta \right) \omega$  

(Some authors define $\delta$ as the negative of this)

Properties:
- $\delta^2 = 0$
- In coordinates:
  \[
  (\delta \omega)_{i_1 \cdots i_p} = \frac{1}{p!} \left( \sum_{k=1}^{p} \omega_{j_1 \cdots j_{p-1} k} \right)_{i_1 \cdots i_p}
  \]
- If $M$ is contractible (and in every contractible part):
  $\delta \omega = 0 \Rightarrow \exists \omega : \omega = \delta \omega$

Exercises:
- Show the above.
- Determine whether or not $\delta$ is a derivation.

Use $d$ and $\delta$ to obtain the Maxwell equations on $M$

Define:

- Electric field
  \[\mathbf{E}(x) = \begin{pmatrix} E_1, 0, -E_2, -E_3 \end{pmatrix}\]
- Field strength:
  \[\mathbf{F}_{\mu\nu}(x) = \begin{pmatrix} \partial_\mu E_\nu - \partial_\nu E_\mu, \partial_\mu B_\nu - \partial_\nu B_\mu \end{pmatrix}\]
- Magnetic field:
  \[\mathbf{B}(x) = \begin{pmatrix} 0, B_3, -B_2, B_1 \end{pmatrix}\]

"Current" 4-form:

\[j(x) = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} j^{\mu} dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma}\]

Then: The Maxwell Equations read:

- "Homogeneous Maxwell equations" (independent of metric)
  \[d \mathbf{F} = 0, \quad \delta \mathbf{F} = \star \mathbf{j}\]
- "In homogeneous cases, Maxwell Equations" (dependent on the metric)

"Current" 4-form, i.e., cotangent vector field
Remarks:

- $F$ is assumed to be an exact 2-form, i.e.,
  \[ F = dA \]
  (the 1-form $A$ is called the 4-potential)

- This already implies the homogeneous Maxwell equations:
  \[ dF = d^2A = 0 \]
  \[ \Rightarrow \text{One calls them "structure equations".} \]

- General relativity also possesses structure equations.

Remark:

The gauge principle of electrodynamics is the observation that, for any $w \in \Lambda_0$:

\[ A \text{ and } \tilde{A} := A + dw \]

describe the same physics.

They do because the (classically) observable fields are only the $E$ and $B$ fields in $F$ and since $d^2 = 0$:

\[ F = dA = d\tilde{A} \]
The Laplacian/d’Alembertian, $\Delta$, $\Box$:

- **Definition of the Laplacian:**
  \[ \Delta := \delta d + d \delta \]

- **Clair:** $\Delta : \Lambda^p(M) \to \Lambda^p(M)$

- If signature $s=1$: Then also called _d’Alembertian_ and denoted $\Box := dS + Sd$.

- **Action on, e.g., $f \in \Lambda_0^1(M)$ in a chart:**
  \[ \Delta f = \frac{1}{\sqrt{|g|}} \left( \nabla^g \cdot \nabla^g f \right) = \left( \frac{-3x^2 + 2x^2 + 2x^2 - x^2}{\sqrt{|g|}} \right) f \]

**Properties of the d’Alembertian, $\Box$ in the Hilbert space $\Lambda(M)$:**

- **Defined:** $\Box : \Lambda_0^p(M) \to \Lambda_0^p(M)$
  \[ \Box : \rightarrow (Sd + dS) \rightarrow \]

- **In the Hilbert space $\Lambda(M)$:**
  \[ \Box = Sd + dS \text{ s.t. } \langle \delta \Box \beta, \beta \rangle = (\delta d, \beta) \]

- $\Box$ is self-adjoint, $\Box = \Box^*$ for suitable boundary conditions, or if $\partial M = \emptyset$ and assuming $(,)$ is positive definite.

- **Exercises:**
  - Verify $\Box = \Box^*$ formally, using only $\delta = -d^*$.
  - Verify that $\Box \ast = \ast \Box$, $\delta d = d \delta$, $\delta S = S \delta$. 
Consequences of the self-adjointness of $\Delta$:

- If $\Lambda$ is a Hilbert space

A) The operators $\Delta$ and $\Box$ can be diagonalized, with real spectrum.

B) For Riemannian manifolds, $\text{spec}(\Delta) \subset [0, \infty)$.

C) For compact Riem. manifolds (of finite volume): $\text{spec}(\Delta)$ is discrete.

D) Then, $\text{spec}(\Delta)$ is carrying a lot of information about $(M, g)$!

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential geometry with functional analysis, i.e., the languages of general relativity and quantum field theory.

Application: Klein-Gordon "action":

$$S[\phi] := \frac{1}{2} \sum_{m} \int_{\Omega} \left( \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} \right) \sqrt{g(x)} \, d^\nu x$$

Recall special relation $\Box g = \nabla \mathbf{1} = \mathbf{0}$.

$$S[\phi] = \int_{M} \left( g^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} - m^2 \phi \right) \sqrt{g} \, d^\nu x$$

$$= -\frac{1}{2} \int_{M} \phi \frac{\partial}{\partial x^\mu} \left( \frac{\partial g}{\partial x^\mu} \phi \frac{\partial g}{\partial x^\nu} \phi \right) \frac{1}{\sqrt{|g|}} \, d^\nu x$$

$$= -\frac{1}{2} \int_{M} \phi (\Box \phi) \Omega$$
Obtain the Klein Gordon wave equation:

Recall: Euler Lagrange equation \( \frac{\partial L}{\partial \phi} - \frac{d}{dx} \left( \frac{\partial L}{\partial \frac{d\phi}{dx}} \right) = 0 \)

Here: \( L = -\frac{1}{2} \Phi \frac{\partial \Phi}{\partial x} \) (the 0-form that we are integrating: \( S = \int \Phi \phi \) )

Obtain Klein Gordon equation:

\[ \Phi = 0 \] (with mass: \( L = -\frac{1}{2} \Phi (\partial^2 \phi) \))

\( \Rightarrow (\partial^2 \phi + m^2) \phi = 0 \)

Q: Which physical fields are described by K-G fields?

A: Meson fields

- Higgs field (gives all particles their mass. Found at LHC. Nobel to Higgs, Englert (2013))

- Inflaton field (crucial ingredient in modern cosmology + see later)