How to describe the "shape" of a manifold?

Historically:

E.g., on a potato-shaped surface:

\[ a^2 + b^2 \neq c^2 \]
\[ d + f + 90^\circ \neq 180^\circ \]

Helmholtz & Hampf already considered checking for curvature of space this way.

Recall:

\[ \text{Defined } g_{\mu\nu}(x) \]
\[ \Rightarrow \text{infinite straight distances} \Rightarrow \text{finite distances} \Rightarrow \text{shape} \]

Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

Example: (view earth from top)

- Start with a vector at the pole.
- Parallel transport down to some lower latitude, along that latitude and then back to pole.
- Vector will arrive at pole rotated, in spite of having only been parallel transported!

This motivates:

Try to define local shape through "derivative" of vectors with respect to parallel transport!
The Covariant Differentiation, $\nabla$:

**Definition:** Any linear map of tangent vector fields

$$\nabla : T'(M) \times T'(M) \to T'(M)$$

Recall: Lie derivative yields rate of change under infinitesimal diffeomorphisms.
It is metric independent.

Recall: $L_g \gamma = [g, \gamma]$

$$L_g \gamma = \left\{ \begin{align*}
\mathcal{L}_g \gamma &= \left[ g, \gamma \right] \\
\mathcal{L}_g \gamma &= \left\{ \gamma \right\}
\end{align*} \right.$$ outskirts

$$\mathcal{L}_g \gamma = \gamma \cdot \frac{\partial}{\partial x^j}$$

$$\mathcal{L}_g \gamma = \left[ g, \gamma \right] = \gamma \cdot \frac{\partial}{\partial x^j}$$

$$(\text{I}) \quad \mathcal{L}_g \gamma = f \frac{\partial}{\partial x^j} \eta, \quad \forall f \in \mathbb{R}(M)$$

$$(\text{II}) \quad \mathcal{L}_g (f \gamma) = \mathcal{L}_g (f) \gamma + f \mathcal{L}_g \gamma$$

In called a covariant derivative or affine connection.

Use of $L_g \gamma$ to find how components of a
tensor change under an
infinitesimal change of
coordinates:

$$x' \to x = x + \xi$$

$$x \to x'$ is infinitesimal diffeomorphism.

Note:

For now, let us assume a metric has not (yet) been specified, so we are free to choose $\nabla$, and this choice defines the shape of $M$!

$\nabla$ in a chart:

- Choose an bases for $T_x(M)$, e.g.: $\left\{ \frac{\partial}{\partial x^j} \right\}$

- Given a covariant derivative $\nabla$, consider its action on basis vectors, such as, e.g.: $\gamma = \frac{\partial}{\partial x^i}, \eta = \frac{\partial}{\partial x^j}$:

$$(\nabla_x \gamma)^{\nu} = \nabla^{\nu}_{\partial x^i} \left( \gamma^{\alpha} \frac{\partial}{\partial x^\alpha} \right) = \nabla_x^{\nu} \gamma^{\alpha} \left( \frac{\partial}{\partial x^\alpha} \right)$$

Recall: $L_{\xi} \gamma = 0$

Thus, via the axioms:

$$\mathcal{L}_g \gamma = \mathcal{L}_g \left( \gamma^\alpha \frac{\partial}{\partial x^\alpha} \right)$$

$$= \gamma^\alpha \mathcal{L}_g \left( \frac{\partial}{\partial x^\alpha} \right) + \frac{\partial}{\partial x^\alpha} \mathcal{L}_g \gamma$$

$$= \left( \gamma^\alpha \mathcal{L}_g \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \mathcal{L}_g \gamma \right)$$

$$= \left( \left[ g, \gamma \right] \cdot \frac{\partial}{\partial x^\alpha} \right)$$

Thus:

$$\mathcal{L}_g \gamma = \mathcal{L}_g \left( \gamma^\alpha \frac{\partial}{\partial x^\alpha} \right)$$

The $\Gamma^\nu_{\alpha \beta}$ are called “Christoffel symbol” or “Connection coefficients”.

Thus, via the axioms:

$$\mathcal{L}_g \gamma = \gamma^\alpha \mathcal{L}_g \left( \frac{\partial}{\partial x^\alpha} \right) + \frac{\partial}{\partial x^\alpha} \mathcal{L}_g \gamma$$

Recall: $\mathcal{L}_g \gamma = \left[ g, \gamma \right]$

$$= \gamma^\alpha \mathcal{L}_g \left( \frac{\partial}{\partial x^\alpha} \right) + \frac{\partial}{\partial x^\alpha} \mathcal{L}_g \gamma$$

$$= \left( \gamma^\alpha \mathcal{L}_g \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \mathcal{L}_g \gamma \right)$$

$$\left[ g, \gamma \right] \cdot \frac{\partial}{\partial x^\alpha}$$

Notation:

$$\gamma^\nu_{\alpha \beta} := \gamma^\nu_{\beta \alpha} + \gamma^\nu \mathcal{L}_g \gamma$$

Thus:

$$\mathcal{L}_g \gamma = \mathcal{L}_g \left( \gamma^\alpha \frac{\partial}{\partial x^\alpha} \right)$$

Thus:

$$\nabla^{\nu}_{\partial x^j} \left( \gamma^{\alpha} \frac{\partial}{\partial x^\alpha} \right)$$

(\ast)
Important: The $\Gamma^k_{ij}$ transform non-tensorially when $x \rightarrow \tilde{x}$:

On one hand:

$$\nabla_{\partial x^c} \frac{\partial}{\partial x^c} = \Gamma^c_{ab} \frac{\partial}{\partial x^c} = \Gamma^c_{ab} \frac{\partial x^k}{\partial x^c} \frac{\partial}{\partial x^k} \quad (I)$$

because $\frac{\partial}{\partial x}$ is tangent vector

On the other hand:

$$\nabla_{\partial x^d} \frac{\partial}{\partial x^d} = \nabla_{\frac{\partial x^i}{\partial x^d}} \left( \frac{\partial x^i}{\partial x^j} \frac{\partial}{\partial x^j} \right) \quad \text{use axiom (b) } \Rightarrow$$

$$= \frac{\partial x^i}{\partial x^d} \nabla_{\frac{\partial x^i}{\partial x^j}} \left( \frac{\partial x^j}{\partial x^d} \frac{\partial}{\partial x^j} \right) \quad \text{use Leibniz rule (c) } \Rightarrow$$

$$= \frac{\partial x^i}{\partial x^d} \left[ \frac{\partial}{\partial x^d} \left( \frac{\partial x^j}{\partial x^d} \right) \frac{\partial x^j}{\partial x^i} + \frac{\partial x^j}{\partial x^i} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right]$$

$$= \left( \frac{\partial}{\partial x^d} \frac{\partial x^i}{\partial x^d} \right) \frac{\partial x^j}{\partial x^i} + \frac{\partial x^i}{\partial x^d} \frac{\partial x^j}{\partial x^i} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad (II)$$

Comparing $I, II \Rightarrow$

$$\Gamma^c_{ab} \frac{\partial x^k}{\partial x^c} = \frac{\partial^2 x^k}{\partial x^a \partial x^b} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \Gamma^k_{ij} \quad \text{(if } \frac{\partial x^c}{\partial x^d} \neq 0)$$

$$\Rightarrow$$

$$\nabla^{\partial x^d} \frac{\partial}{\partial x^d} = \frac{\partial x^i}{\partial x^d} \frac{\partial}{\partial x^i} \Gamma^k_{ij} + \frac{\partial x^i}{\partial x^d} \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} \Gamma^k_{ij}$$

This term is indep. of $\Gamma$ only this term would be there, if the $\Gamma^k_{ij}$ were tensor coefficients.

$\Rightarrow \Gamma$ can be seen in one coordinate system and两者 in another.

(can be shown to be equivalent)

Physicists' definition of $\nabla$: Any set of $n^3$ functions $\Gamma^k_{ab}(x)$ which transform this way are defining a covariant derivative $\nabla$. 
The "absolute" covariant derivative $\nabla$:

Consider the covariant derivative but without choosing a direction vector $\xi$:

$$\nabla : T_x(M)^1 \to T_x(M)^1, \quad \nabla : \gamma = \gamma^k \partial_{x^k} \to \nabla \gamma(x) = \gamma^k(x) dx^i \otimes \frac{\partial}{\partial x^k}$$

Indeed: The contraction of $\nabla \gamma$ with $\xi$ yields:

$$\nabla \gamma(\xi) = \gamma^k \partial_{x^k} (\xi^i) \frac{\partial}{\partial x^k} = \gamma^k \xi^i \frac{\partial}{\partial x^k} = \nabla_{\xi} \gamma$$

We defined $\nabla$ algebraically. Now, extract the geometric meaning of $\nabla$:

**Definition:** Assume $\nabla$ is given. Choose a path $\gamma : \mathbb{R} \to M$.

Then, a tangent vector field $\gamma$ is called auto-parallel along $\gamma$, if

$$\nabla_{\dot{\gamma}} \gamma = 0$$

i.e. if $\gamma$ doesn't change under parallel transport along the path $\gamma$.

Note: We'll see that they always exist, i.e., we can always parallel transport a vector a finite distance.
In a chart, 
\[ \eta = \eta^i(x) \frac{\partial}{\partial x^i} \]

and
\[ \gamma : [a, b] \to M \]
\[ \gamma : t \to x^i(t) \]

and the tangent vector:
\[ \frac{d\gamma^i}{dt} = \frac{dx^k}{dt} \frac{\partial}{\partial x^k} \]

Thus:
\[ \nabla_{\gamma^i} \eta = \nabla_{dx^k} \left( \eta^i \frac{\partial}{\partial x^k} \right) = \frac{dx^k}{dt} \nabla_{x^k} \left( \eta^i \frac{\partial}{\partial x^i} \right) \]
\[ = \frac{dx^k}{dt} \left( \frac{\partial \eta^i}{\partial x^k} + \eta^j \Gamma^i_{jk} \frac{\partial x^j}{\partial x^i} \right) \]
\[ = \left( \frac{dx^i}{dt} + \eta^j \frac{dx^k}{dt} \Gamma^i_{jk} \right) \frac{\partial}{\partial x^i} = 0 \]

\[ \Rightarrow \eta \text{ autoparallel to } \gamma \text{ means:} \]
\[ \frac{d\gamma^i}{dt} + \eta^i \frac{dx^k}{dt} \Gamma^i_{jk} = 0 \]

J.e. this is the condition for the vectors of \( \eta \) being parallel translates of each other along \( \gamma \).

Conclusion:

This is 1st order ODEs for \( \eta \). Thus:

Initial condition \( \gamma(\gamma(0)) \Rightarrow \text{solution } \gamma(\gamma(0)) \text{ exists at least locally} \)

\[ \Rightarrow \text{ Proposition:} \]

Given a path \( \gamma : [t, s] \to M \), the autoparallel transport of a tangent vector \( \eta \) at \( \gamma(t) \) to \( \gamma(s) \) is unique.
I.e., the path \( \gamma \) defines a parallel transport map \( \tau \):

\[
\tau(t, s): \quad T_{\gamma(s)} \to T_{\gamma(t)}
\]

\[
\tau(t, s): \quad \eta(\gamma(s)) \to \eta(\gamma(t))
\]

**Q:** Can one use \( \tau \) to obtain \( \nabla \) as a Newton-Leibnitz limit?

**Proposition:** (for the proof, see e.g. the text by Stromme)

\[
\nabla_{\gamma} \eta(\gamma(t)) = \frac{d}{ds} \bigg|_{s=t} \tau(s, t)(\eta(\gamma(s)))
\]

**Note:** Since we can choose paths with arbitrary \( \gamma \), this equation can be used as a geometric definition of \( \nabla \).

\[\nabla\text{ for arbitrary tensors:}\]

- The parallel transport map \( \tau(s, t) \) transports tangent vectors \( \eta \) from \( \gamma(s) \) to \( \gamma(t) \).

**Definition:** \( \tau(s, t) \) also parallel transports the dual vectors \( \omega \), namely so that contraction is conserved:

\[
\tau(\omega)(\tau(\eta)) = \omega(\eta) \quad (C)
\]

**Extension of \( \tau \) to tensor products:**

\[
\tau(s_1 \otimes s_2) := \tau(s_1) \otimes \tau(s_2) \quad (T)
\]

\( s_1 \) and \( s_2 \) are tensors of arbitrary rank.
**Definition:**

\[ \nabla_S S'(p) := \left. \frac{d}{dt} \tau(t,0)(S'(x(t))) \right|_{t=0} \]

where \( x \) is any path through \( p \) obeying:

\[ x(0) = \xi(p), \quad x(\theta) = p \]

**Absolute covariant derivative:**

\[ \left( \frac{d}{dt} \right)_{x(t)} \tau(t,0)(S'(x(t))) := \nabla_S S'(x_1, \ldots, x_n, \omega_1, \ldots, \omega_k, S) \]

**Properties of \( \nabla \):**

\* \( \nabla \) is a derivation:

\[ \nabla_S (S \otimes S_2) = \left. \frac{d}{ds} \right|_{s=\xi} \tau(S \otimes S_2) = \left. \frac{d}{ds} \right|_{s=\xi} \tau(S) \otimes \tau(S_2) \]

\[ = \left. \frac{d}{ds} \right|_{s=\xi} \tau(S_1) \otimes \tau(S_2) + \left. \frac{d}{ds} \right|_{s=\xi} \tau(S) \otimes \frac{d}{ds} \left. \right|_{s=\xi} \tau(S) \]

\[ = (\nabla_S S_1) \otimes S_2 + S_1 \otimes \nabla_S S_2 \quad (A) \]

\* Eq. (A) implies that \( \nabla \) and contractions do commute.
Action of $\nabla$ on tensors in a chart?

- Recall: $\nabla_{\xi} \frac{2}{\partial x^i} = \xi^\ell \Gamma_{\ell i}^{\kappa} \frac{2}{\partial x^\kappa}$

- Problem: Find $\nabla_{\xi} dx^i = ?$

Consider $\eta \otimes \omega$.

- Differentiate:
  \[ \nabla_{\xi} (\eta \otimes \omega) = (\nabla_{\xi} \eta) \otimes \omega + \eta \otimes \nabla_{\xi} \omega \]

Contrast: (use that $\nabla_{\xi}$ and contraction commute)

\[ \nabla_{\xi} (\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta) \] (※)

(\text{i.e., } \nabla(\omega(\eta)) = \omega(\nabla_{\xi} \eta) + (\nabla_{\xi} \omega)(\eta))

⇒ An expression for $\nabla_{\xi} (\omega(\eta))$:

\[ (\nabla_{\xi} \omega)(\eta) = \omega(\nabla_{\xi} \eta) - \omega(\nabla_{\xi} \eta) \]

Now: Choose $\omega := dx^i$ and $\eta := \frac{2}{\partial x^i}$.

(recall: $\langle \omega, \xi \rangle$ in evaluation of covector on vector, and inner product in general)

⇒ $\nabla_{\xi} dx^i \frac{2}{\partial x^i} = \xi^\ell \langle dx^\ell, \frac{2}{\partial x^i} \rangle - \langle dx^i, \xi^\ell \frac{2}{\partial x^i} \rangle$

= $-\langle dx^i, \xi^\ell \Gamma_{\ell i}^{\kappa} \frac{2}{\partial x^k} \rangle$

= $-\xi^\ell \Gamma_{\ell i}^{\kappa} \frac{2}{\partial x^k}$

⇒ $\nabla_{\xi} dx^i = -\xi^\ell \Gamma_{\ell i}^{\kappa} dx^i$
For general tensors: (by exactly same strategy as above but applied to multiple tensor products, we obtain:

\[ \nabla_3 S(\eta_1, ..., \eta_r, \omega_1, ..., \omega_s) \quad \text{(as in Eq. (\star) above)} \]

\[ = \xi(S'(\eta_1, ..., \eta_r, \omega_1, ..., \omega_s)) \]

\[ - S'(\partial_3 \eta_1, ..., \partial_3 \eta_r, \omega_1, ..., \omega_s) - \ldots \]

\[ - S'(\eta_1, ..., \partial_3 \eta_r, \omega_1, ..., \omega_s) \]

\[ - S'(\eta_1, ..., \eta_r, \partial_3 \omega_1, \omega_2, ..., \omega_s) + \ldots \]

\[ - S'(\eta_1, ..., \eta_r, \omega_1, ..., \partial_3 \omega_s) \]

Choosing the basis vectors \( dx^i \) and \( \frac{\partial}{\partial x^i} \), we obtain for

\[ S = S_{\cdot i_1 ... i_p}^i \frac{\partial}{\partial x^{i_1}} \otimes ... \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p} \]

that \( \nabla_3 S' \) reads

\[ \nabla_3 S' = \xi \Sigma_{\cdot i_1 ... i_p}^{i_1 ... i_p} \frac{\partial}{\partial x^{i_1}} \otimes ... \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p} \]

with:

\[ S_{\cdot i_1 ... i_p}^{i_1 ... i_p} := S_{\cdot i_1 ... i_p}^{i_1 ... i_p} + \Gamma_{\cdot k_1 ... k_p}^{i_1 ... i_p} S_{\cdot j_1 ... j_k}^{j_1 ... j_k} + \ldots + \Gamma_{\cdot k_1 ... k_p}^{i_1 ... i_p} S_{\cdot j_1 ... j_k}^{j_1 ... j_k} - \Gamma_{\cdot k_1 ... k_p}^{i_1 ... i_p} S_{\cdot j_1 ... j_k}^{j_1 ... j_k} - ... - \Gamma_{\cdot k_1 ... k_p}^{i_1 ... i_p} S_{\cdot j_1 ... j_k}^{j_1 ... j_k} \]
Special cases:

- Tangent vector fields:
  \[ \xi_{jk} = \xi_{j,k} + \xi^i \Gamma_{jk}^i \]

- Cotangent vector fields:
  \[ \omega_{ijk} = \omega_{i,jk} - \omega_{j,k} \Gamma_{ki} \]

**Recall:** Specifying \( \triangledown \) specifies parallel transport of vectors and this should specify the manifold's shape, but how?

\[ \Rightarrow \quad \text{Indeed, \( \triangledown \) specifies Torsion & Curvature} \]

**Definition:**

Assume \( \xi, \xi_2 \) and \( \xi_3 \) are tangent vector fields.

- The **torsion** map is defined as:
  \[ T : \xi, \xi_2 \rightarrow T(\xi, \xi_2) := \nabla_{\xi_2} \xi - \nabla_{\xi} \xi_2 - [\xi, \xi_2] \]

  \[ \text{idea: nonzero if small parallelograms don't close, i.e. commutator} \]

- The **curvature** map is defined as:
  \[ R : \xi, \xi_2, \xi_3 \rightarrow R(\xi, \xi_2, \xi_3) \xi_3 = (\nabla_{\xi_2} \nabla_{\xi_3} - \nabla_{\xi_3} \nabla_{\xi_2}) \xi_3 \]