Recall: So far, we have 2 ways to capture shape:

- Specified $g \Rightarrow$ specified distances in $M$
  $\Rightarrow$ implicitly specified "shape" of $M$

(Notice (for essay): See also my new paper 1510.02725)

Then, new:

- Specified $\triangledown \Rightarrow$ specified parallel transport in $M$
  $\Rightarrow$ implicitly specified "shape" of $M$

Question:

How does $\triangledown$ determine "shape"? Through:

**Torsion & Curvature**

Recall:

$$\Gamma^{\tau}_{\alpha\beta} = \frac{\partial x^\tau}{\partial x^\alpha} \frac{\partial x^\tau}{\partial x^\beta} \Gamma^k_{\tau j} + \frac{\partial x^\tau}{\partial x^\alpha} \frac{\partial^2 x^k}{\partial x^\beta \partial x^\tau}$$

Notice: The antisymmetric part of $\Gamma$ transforms tensorially!

$$\begin{cases} 
\Gamma^{\tau}_{(\text{sym})\tau j} := \frac{1}{2} \left( \Gamma^k_{\tau j} + \Gamma^k_{\tau i} \right) \\
\Gamma^{\tau}_{(\text{asym})\tau j} := \frac{1}{2} \left( \Gamma^k_{\tau j} - \Gamma^k_{\tau i} \right)
\end{cases}$$

$$\Rightarrow \Gamma^{(\text{asym})\tau}_{\alpha\beta} = \frac{\partial x^\tau}{\partial x^\alpha} \frac{\partial x^\tau}{\partial x^\beta} \Gamma^k_{(\text{asym})\tau j}$$

Definition: $J^{\tau}_{\alpha\beta} := 2 \Gamma^{\tau}_{(\text{asym})\tau j}$ in the "Torsion tensor"

(Notice: Since $\Gamma$ is not a tensor, but $\Gamma_{(\text{sym})}$ is, $\Gamma_{(\text{sym})}$ is not a tensor)
In General Relativity: one assumes torsionless $\nabla$, i.e.: $\mathbf{T} = 0$.

**Idea:** "(Extended) equivalence principle:"

Christoffel $\Gamma$ will express gravitational and pseudo forces. Therefore, we require that around each $\mathbf{p} \in \mathcal{M}$ there exists a chart so that $\Gamma(\mathbf{p}) = 0$ (i.e., no such forces in this field).

This rules out the existence of torsion:

**Why?** The torsion is a tensor.

$\Rightarrow$ $\mathbf{T}$ transforms linearly with invertible Jacobian matrix

$$\check{T}_{j k}^i(x) = \frac{\partial x^i}{\partial x^a} \frac{\partial x^b}{\partial x^j} \frac{\partial x^c}{\partial x^k} T_{a b c}(x)$$

$\Rightarrow$ If $T_{j k}$ vanishes in one chart, it vanishes in all charts.

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**Proposition:**

Vice versa, if $T_{j k}^i(x) = 0 \forall x \in \mathcal{M}$,

then there is for every $\mathbf{p} \in \mathcal{M}$ a chart with $\Gamma_{j k}^i(\mathbf{p}) = 0$.

**Recall:** $\xi$ is autoparallel to a path $\gamma: t \rightarrow x(t)$ if

$$\nabla_t \xi = 0$$

Meaning: $\xi$ is parallel transported along the path $\gamma$ in $\mathcal{M}$.

Explicitly:

$$\frac{d\xi^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} \xi^j = 0$$

**Geodesics:** A curve $\gamma: t \rightarrow x(t)$ is called a geodesic

if $\xi$ is autoparallel along $\gamma$, i.e., if

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

Meaning: The path $\gamma$ in $\mathcal{M}$ is such that the path's tangent vectors are parallel translates of each other.
In charts, geodesics $x^r(t)$ obey:

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \quad (\ast)$$

**Theory of ordinary differential equations:**

- Given $p = x(0)$, each initial condition $\xi = x(0)$ belongs to a unique geodesic $\gamma_\xi$ of nonzero length.

**Notice:** If $\gamma_\xi(t)$ solves $(\ast)$ then $\gamma_\xi(\lambda t)$ also solves $(\ast)$ and for $\lambda \in \mathbb{R}$:

$$\gamma_{\xi}(t) = \gamma_{\xi}(\lambda t) \quad (\ast)$$

*(Exercise: verify)*

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"Exponential map:"

- Consider a fixed point $p \in M$.

The exponential map is defined through:

$$\exp_p : T_p(M) \to M \quad \text{(really from a neighborhood of 0 in } T_p(M) \text{ to a neighborhood of } p \in M)$$

$$\exp_p : \xi \to \exp_p(\xi) = \gamma_{\xi}(1)$$

where $\gamma$ is the geodesic with $\gamma(0) = p$, $\gamma(0) = \xi$.

**Observe:**

From $(\ast)$ we obtain:

$$\gamma_{\xi}(\lambda) = \gamma_{\xi}(1) = \exp_p(\lambda \xi) \quad (\ast)$$
"Geodesic" or "Riemann normal" coordinates:

\[ \exp_p \text{ is a diffeomorphism from a neighborhood of } 0 \in T_p(M) \cong \mathbb{R}^m \text{ into a neighborhood of the point } p \in M. \]

\[ \Rightarrow \exp_p \text{ provides a chart around } p: \]

- Choose a basis, say \( e_1, e_2, \ldots, e_m \) of \( T_p(M) \), then:
  \[ \xi = \xi^i e_i. \]

- Through \( \exp_p \), the \( \xi^i \) become the coordinates of points in a neighborhood of \( p \in M: \)
  \[ (\xi^1, \ldots, \xi^m) \rightarrow \exp_p(\xi^i e_i) \in M \]

- These \( \xi^i \) are called "normal" or "geodesic coordinates."

\[ \Rightarrow \text{Geodesics, } y, \text{ through } p \text{ are straight lines in a normal chart about } p! \]

- Recall (E):

\[ y^\mu(\lambda) = \exp_p(\lambda \xi^i e_i) \]

- For varying \( \lambda \), one moves along the geodesic path.

- Thus: In geodesic charts, geodesics through \( p \) are straight lines of constant velocity \( \xi^i \).

- Does this mean \( \Gamma^\mu_{\nu i}(p) = 0 \)? No!
Geodesic eqn. at $p$: 
\[ \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(p) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \]

Thus: 
\[ \left( \Gamma^k_{ij}(p) + \Gamma^k_{ji}(p) \right) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \]

$\Rightarrow \Gamma^k_{ij}(p) = 0$ in geodesic coords.

\[ \Rightarrow \text{Indeed: If the torsion vanishes, } \Gamma^k_{ij}(p) = \frac{1}{2} \mathcal{J}^k_{ij}(p) = 0 \]

then for each $p \in M$ there exists a chart in which the entire gravity and pseudo force field vanishes at $p$.

Note: Quantum fluctuations may induce torsion!

So, let’s nonetheless ask:

What would torsion mean, geometrically?

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Abstract definition of Torsion:

- Assume $\xi_1$ and $\xi_2$ are tangent vectors at $p \in M$:

Then, the Torsion map is defined as:

\[ \mathcal{J} : T_p^1(M) \times T_p^1(M) \to T_p^1(M) \]

\[ \mathcal{J} : \xi_1, \xi_2 \to \mathcal{J}(\xi_1, \xi_2) := \xi_1 \xi_2 - \xi_2 \xi_1 - [\xi_1, \xi_2] \]

- It is used to define the Torsion tensor, $\mathcal{J}$, through:

\[ \mathcal{J} \in T_p^2(M) \]

\[ \mathcal{J}(\omega, \xi_1, \xi_2) = \omega(\mathcal{J}(\xi_1, \xi_2)) \in \mathbb{R} \]

\[ \text{we could also write: } \mathcal{J}(\omega, \xi_1, \xi_2) = \omega \quad \text{contraction yields a number} \]

\[ \text{feeding $3$ conds into $2$ nums } \rightarrow \mathcal{J}(\omega, \xi_1, \xi_2) \in T_p^1(M) \]
Compare with prior definition:

- Choose canonical bases \(\omega := dx^k\), \(\xi_1 := \frac{2}{\partial x^i}\), \(\xi_2 := \frac{2}{\partial x^j}\):

\[
\mathcal{J}_{ij}^k := dx^k(\mathcal{J}(\frac{2}{\partial x^i}, \frac{2}{\partial x^j}))
= \left< dx^k, \mathcal{J}(\frac{2}{\partial x^i}, \frac{2}{\partial x^j}) \right>
= \left< dx^k, \nabla_{\xi_1} \frac{2}{\partial x^i} - \nabla_{\xi_2} \frac{2}{\partial x^j} - \left[ \frac{2}{\partial x^i}, \frac{2}{\partial x^j} \right] \right>
\]

Recall \(\frac{2}{\partial x^i} \nabla_{\xi_1} \frac{2}{\partial x^i} = \mathcal{J}_{ij}^k \frac{2}{\partial x^j} \mathcal{J}(\frac{2}{\partial x^i}, \frac{2}{\partial x^j}) f = 0 \quad \forall f \)

\[
= \left< dx^k, \Gamma_{ij}^r \frac{2}{\partial x^i} - \Gamma_{ji}^r \frac{2}{\partial x^j} \right> = \Gamma_{ij}^r \delta^k_r - \Gamma_{ji}^r \delta^k_i
\]

\(\Rightarrow \quad \mathcal{J}_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k\)

Geometric meaning of torsion? Parallelograms would not close!

Travel from \(p\) infinitesimally in \(\xi\) and then \(\gamma\) direction, and compare with the reverse. (In flat space: \(x^k + \xi^k + \gamma^k = x^k + \xi^k + \gamma^k\))

\[\begin{align*}
\xi, \gamma \in T_p^1 & \quad \xi, \gamma \in T_p^1 \\
\tilde{\xi} \in T_\gamma & \quad \tilde{\gamma} \in T_\gamma
\end{align*}\]

Recall parallel transport: \(\nabla_{\xi} v = 0\)

\[
\frac{dv^k}{dt} + \Gamma_{ij}^k \frac{dx^i}{dt} v^j = 0
\]

\[\tilde{\xi}(t) = ?\]

\[
\tilde{\xi}^k(x^i + \gamma^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) \quad \text{Now use } v = \xi, \frac{dx^i}{dt} = \gamma^i:
= \xi^k(x^i) - \Gamma(x)^k_{ij} \gamma^j \xi^i
\]

\(\Rightarrow\) Cds. of \(\overline{uv}: \quad x^a + \xi^a + \gamma^a = \Gamma(x)^a_{ij} \gamma^i \xi^j\)
Analogously obtain: \(\text{cds. of } u: x^a + \xi^a + \eta^a - \Gamma'(x)^{a}_{ij} ; \hat{\xi}^j \hat{\eta}^i \)

\(\Rightarrow \text{cd. distance from } u \text{ to } \hat{u} \text{ is: } \left( \Gamma'(x)^{j}_{ij} - \Gamma'(x')^{j}_{ij} \right) \hat{\xi}^i \hat{\eta}^j = \mathcal{T}^i_{ji} \hat{\xi}^i \hat{\eta}^j \)

\text{Comment: We had:}

\[
\hat{\xi}^k(x'^i + \xi^i) \approx \xi^k(x'^i) + \frac{d\xi^k}{dx^i}(x^i) = \xi^k(x'^i) - \Gamma'(x)^k_{ij} \xi^j \xi^i
\]

This in also:

\[
= \xi^k(x'^i) - \left( \xi^j \xi^i_{,j} + \Gamma'(x)^k_{ij} \xi^j \xi^i \right) + \xi^j \xi^i_{,i}
\]

\[
= \xi^k(x'^i) - \xi^j \xi^i_{,j} + \xi^j \xi^i_{,i}
\]

Thus: cd distance from \( u \) to \( \hat{u} \) is:

\[
(x'^a + \xi^a + \eta^a - \xi^i \xi^j_{,i} + \xi^j \xi^i_{,i}) - (x^a - \xi^a + \xi^i \xi^j_{,i} - \xi^j \xi^i_{,i}) = \mathcal{T}^i_{ji} \hat{\xi}^i \hat{\eta}^j
\]

Recall that indeed: \( \mathcal{T}^i_{ji} \hat{\xi}^i \hat{\eta}^j \rightarrow \mathcal{T}(\gamma, \xi) = \nabla_\gamma \xi - \nabla_\xi \gamma - [\gamma, \xi] \)

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**Curvature:**

Assume \( \xi_1, \xi_2 \) and \( \xi_3 \) are tangent vectors at p.c.m.

- The curvature map, \( R \), is defined through:

\[
\mathcal{R} : \xi_1, \xi_2, \xi_3 \rightarrow \mathcal{R}(\xi_1, \xi_2)\xi_3 = (\nabla_\xi_1 \xi_2 - \nabla_\xi_2 \xi_1 - \xi_1 \mathcal{L} \xi_2)\xi_3
\]

- It defines the curvature tensor, \( R \),

\[
\mathcal{R} \in T^3_j(M)
\]

through:

\[
\mathcal{R}(\omega, \xi_1, \xi_2, \xi_3) = \langle \omega, \mathcal{R}(\xi_1, \xi_2)\xi_3 \rangle \in \mathbb{R}
\]
In a chart:

\[ R^i_{jk} = \left\langle dx^i, R \left( \frac{\partial^2}{\partial x^j \partial x^k}, \frac{\partial^2}{\partial x^k \partial x^i} \right) \right\rangle \]

\[ = \left\langle dx^i, \left( \sum_{\alpha} \frac{\partial^2}{\partial x^\alpha \partial x^j} \frac{\partial^2}{\partial x^k \partial x^\alpha} - \sum_{\alpha} \frac{\partial^2}{\partial x^\alpha \partial x^i} \frac{\partial^2}{\partial x^k \partial x^\alpha} \right) \right\rangle \]

\[ = \left\langle dx^i, \left( \sum_{\alpha} \frac{\partial^2}{\partial x^\alpha \partial x^j} \frac{\partial^2}{\partial x^k \partial x^\alpha} - \sum_{\alpha} \frac{\partial^2}{\partial x^\alpha \partial x^i} \frac{\partial^2}{\partial x^k \partial x^\alpha} \right) \right\rangle \]

\[ = \left\langle dx^i, \right\rangle \left( \sum_{\alpha} \left( \Gamma^{s}_{ij,k} + \Gamma^{s}_{ki,j} \Gamma^{s}_{j,k} - \Gamma^{s}_{kj,i} \Gamma^{s}_{i,k} - \Gamma^{s}_{ij,k} \Gamma^{s}_{i,k} \right) \frac{\partial^2}{\partial x^{s}} \right) \]

\[ = \left\langle dx^i, \right\rangle \left( \sum_{\alpha} \left( \Gamma^{s}_{ij,k} + \Gamma^{s}_{ki,j} \Gamma^{s}_{j,k} - \Gamma^{s}_{kj,i} \Gamma^{s}_{i,k} - \Gamma^{s}_{ij,k} \Gamma^{s}_{i,k} \right) \frac{\partial^2}{\partial x^{s}} \right) \]

\[ = \left\langle dx^i, \right\rangle \left( \sum_{\alpha} \left( \Gamma^{s}_{ij,k} + \Gamma^{s}_{ki,j} \Gamma^{s}_{j,k} - \Gamma^{s}_{kj,i} \Gamma^{s}_{i,k} - \Gamma^{s}_{ij,k} \Gamma^{s}_{i,k} \right) \frac{\partial^2}{\partial x^{s}} \right) \]

(At origin of geodesics do they vanish ?)

Curvature tensor’s meaning?

Intuition:

- Contains derivation of \( \Gamma \) \( \Rightarrow \)
- expresses variation in pseudo and gravitational forces \( \Rightarrow \)
- expresses the strength and direction of “tidal forces”\( \Rightarrow \)

Geometry:

- Curvature expresses noncommutativity of two parallel transports, namely:
Proposition: (Ricci Identity)

Assume the torsion vanishes and that $\xi$ is a vector field. Then:

$$\xi^a_{\ jcd} - \xi^a_{\ jdc} = R^a_{\ cdb} \xi^b$$

(here: $\xi^a_{\ jcd} = \xi^a_{\ jcd\ d}$ etc.)

Remark:

(Aside: easy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:

$$(\xi - \xi)_{ij} \approx \eta^b \eta^c R^a_{\ cdb} \xi^a$$

Proof of Ricci identity:

- Assume $\xi, \eta, \nu$ are vector fields.

- Then, $R(\xi, \eta)\nu := \nabla_x (\nabla_\eta \nu) - \nabla_\eta (\nabla_x \nu) - \nabla_{[\xi, \eta]} \nu$ reads

  $$R^a_{\ cdb} \xi^b \eta^c \nu^d = (\nu_j d \eta^d)_{jc} \xi^c - (\nu_j d \xi^d)_{jc} \eta^c - \nu_j d (\xi^d \eta^c - \xi^c \eta^d)$$

  used Torsion: $T(\xi, \eta) := \xi \eta - \eta \xi - [\xi, \eta] = 0$ i.e. $[\xi, \eta] = 0$

Terms cancel:

$$\Rightarrow R^a_{\ cdb} \xi^b \eta^c \nu^d = (\nu_j d \nu^d)_{jc} \xi^c$$

- True $\forall \xi, \eta$ $\Rightarrow R^a_{\ cdb} \nu^d = \nu_{jcb} - \nu^a_{jbc}$

The "Bianchi Identities":

- They are automatic relations among torsion and curvature, by construction.

**Preparation:** □ for maps!

Consider an arbitrary $\mathcal{F}(M)$-linear map:

$$K : \mathfrak{X}_1 \times \mathfrak{X}_2 \times \ldots \times \mathfrak{X}_r \rightarrow K(\mathfrak{X}_1, \ldots, \mathfrak{X}_r)$$

Tangent vector

Tangent vector

i.e. at each $p \in M$:

$$K : T_p(M)^r \rightarrow T_p(M)^r$$

- We can view $K$ as a tensor $\tilde{K} \in T_p(M)^r$, namely:

$$\tilde{K}(\omega, \mathfrak{X}_1, \ldots, \mathfrak{X}_r) := \langle \omega, K(\mathfrak{X}_1, \ldots, \mathfrak{X}_r) \rangle$$

- Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map $K$:

$$\langle \omega, (\nabla \tilde{K})(\mathfrak{X}_1, \ldots, \mathfrak{X}_r) \rangle := \nabla \tilde{K}(\omega, \mathfrak{X}_1, \ldots, \mathfrak{X}_r)$$

New concept: covariant derivative of a map $K : T_p(M)^r \rightarrow T_p(M)^r$ when fed one covariant vector.

Using □ for map:
1st Bianchi Identity:
\[ \sum_{\text{cyclic}} R(\xi, \gamma) \nu = \sum_{\text{cyclic}} \left( \nabla_\xi (\nabla_\gamma \nu) + \nabla_\gamma (\nabla_\xi \nu) \right) \]

2nd Bianchi Identity:
\[ \sum_{\text{cyclic}} \left( (\nabla_\xi R)(\gamma, \nu) + R(\nabla_\xi (\gamma, \nu)) \right) = 0 \]

with obvious simplification in case \( \nabla = 0 \).

Note: They are automatically obeyed equations, just like any set of linear operators obeys the Jacobi identity with respect to \( \nabla \). Indeed that's why:

Proof of 1st Bianchi: (assuming no torsion)
\[ \sum_{\text{cyclic}} R(\xi, \gamma) \nu = 0 \]

Indeed:
\[ \nabla_\xi (\nabla_\gamma \nu - \nabla_\nu \gamma) - \nabla_{[\xi, \gamma]} \nu + \text{cyclic} \]

[skip by 1 cyclically, skip by 1 cyclically]
\[ = \nabla_\xi (\nabla_\gamma \nu - \nabla_\nu \gamma) - \nabla_{[\xi, \gamma]} \nu + \text{cyclic} \]

Exercise: Prove that, without torsion:
\[ \nabla_\gamma \nu - \nabla_\nu \gamma = [\gamma, \nu] \quad (\text{Cov}) \]

\[ = \nabla_\xi [\gamma, \nu] - \nabla_{[\xi, \gamma]} \nu + \text{cyclic} \]

because again \( \nabla_\xi \nu - \nabla_\nu \xi = [\xi, \nu] \)
Recall:

Assume $A, B, C$ are linear maps $V \rightarrow V$

Then: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$

i.e., the Jacobi identity holds.

Proof: Simply spell out the commutators.

Remark: This means that, e.g., in quantum mechanics, all objects that need to be representable as operators on the Hilbert space must obey the Jacobi identity, e.g., generators of symmetries.