Recall: If we choose the bases $\{\frac{2}{\sqrt{\gamma}} \delta x^\mu \}$, $\{dx^\nu\}$, then:

$$E_{\mu} = L_{\text{EM}} = -\frac{1}{16\pi} F^{\mu
u} F_{\nu\nu}$$

$$S'[g_{\mu\nu}, \psi_{(\nu)}] = \int \left( \frac{1}{16\pi G} R(g_{\mu\nu}(x)) + L_{\text{matter}}(g_{\mu\nu}(x), \psi_{(\mu)}, \psi_{(\nu)}(x)) \right) \sqrt{\gamma} d^4 x$$

$$\frac{\delta S'}{\delta \psi_{(\nu)}} = 0 \quad \Rightarrow \quad \text{Eqs. of motion of matter}$$

(Maxwell, Klein Gordon eqns. etc)

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad \text{Einstein equations:}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

What is the Einstein equation when using a frame so that $g_{\mu\nu}(x) = \gamma_{\mu\nu}$?

Recall:

- Frames $\{\Theta^\mu\}, \{e_{\mu}\}$:

Often, one uses as the bases of $T_p(M)$ and $T_p(M)^*$, the canonical bases $\{dx^\mu\}$ and $\{\frac{2}{\sqrt{\gamma}} \delta x^\mu\}$ respectively, which suggest themselves when one chooses coordinates, say $(x^0, ..., x^3)$. Thus, when changing coordinate system, $x \rightarrow \tilde{x}$, one also usually automatically changes basis in $T_p(M), T_p(M)^*$. 
Important: The only reason why the components of a tensor can change when we change coordinates is that we can change basis in the (co-)tangent spaces, namely from one canonical basis to another canonical basis, when we change coord. system.

Recall: \( \frac{\partial}{\partial x^2} = \frac{\partial}{\partial y^2} \rightarrow \frac{\partial}{\partial x^2} = \frac{\partial}{\partial y^2} \)

We notice: If we choose a fixed basis, say \( \{ e_\mu^\nu \} \), then the coefficients of tensors no longer depend on the choice of coordinates!

E.g.: \( \xi = \xi^\nu e_\nu \)

Conversely: Even staying with one coordinate system, we can freely change our choice of basis's in the (co-)tangent spaces:

\[\Theta^\nu = A^\nu_\mu \Theta^\mu\]
\[e'_\mu = (A^\nu_\mu) e_\nu\]

So we have e.g.:

\[\xi = \xi^\nu e_\nu = \xi^\nu A^\nu_\mu e'_\mu = \xi^\nu e'_\mu\]

i.e.:

\[\xi^\nu = A^\nu_\mu \xi'\]

Examples:

\[\Omega^\nu_\mu = A^\nu_\sigma (A^{-1})^\sigma_\beta \Omega^\beta_\mu\]

But: the connection form \( \omega^\nu_\mu(s) = \xi^k \Gamma^\nu_\mu_k \) obeys:

\[\omega^\nu_\mu = A^\nu_\sigma \omega^\sigma_\nu (A^{-1})^\mu_\beta - (dA)^\nu_\sigma (A^{-1})^\mu_\beta\]
How to specify frames?

In an arbitrary coordinate system, we may specify the bases in terms of the canonical bases:

$$\theta^i(x) = A^i_j(x) \, dx^j$$

Note: the $A^i_j(x)$ change non-trivially when changing the coordinate system!

Our choice now: orthonormal frames, or "Tetradss":

- We say that a frame $\{\theta^i, e_i\}$ is orthonormal if in their frame, for all $p \in M$:

$$g(e_i, e_j) = \delta_{ij} \Rightarrow \eta_{\mu\nu} = \theta^i \theta^j \eta_{\mu\nu} \Rightarrow \eta_{\mu\nu}^{ij} = \gamma_{\mu\nu}$$

- Existence? Always: At each $p \in M$ may choose e.g. $\theta^i = dx^i$ where $dx^i$ are canonical ON basis at centre of a geodesic chart.

- Uniqueness?

For a given space-time, $(M, g)$, any ON frame yields a new ON frame by transforming the bases through:

$$\theta'^i(x) = \Lambda(x)^i_j \cdot \theta^j(x),$$

if the linear maps $\Lambda(x)$ preserve the orthogonality:

$$\eta_{\mu\nu} \theta'^i \theta'^j = \eta_{\mu\nu} \Rightarrow \eta_{\mu\nu} = \gamma_{\mu\nu}$$

i.e. if:

$$\Lambda^a \Lambda^b \eta_{\mu\nu} = \gamma_{\mu\nu}$$

$\Rightarrow$ Frames are unique up to local Lorentz transformations.
Re-express the degrees of freedom:

- We used to specify space-times through these data: \((M, g)\)
- Now, let us specify space-times, equivalently, through data \((M, \Theta^\nu)\):

**Namely:**
Assume the \(\Theta^\nu\) are given w. r. t. a basis \(\{dx^\nu\}\), through functions \(A^\nu\)
\[\Theta^\nu(x) = A^\nu_a(x)dx^a\]

so that:
\[g_{\mu \nu} = \Theta^\mu \Theta^\nu = \eta_{\mu \nu} A^\mu_a(x)dx^a \otimes dx^b = g_{\mu \nu}(x)dx^\mu \otimes dx^\nu\]

**Notice:** knowing the \(A^\nu_a(x)\), we can reconstruct \(g_{\mu \nu}(x)\) in basis \(\{dx^\nu\}:

We use that the abstract \(g\) is the same in every basis:
\[g = \eta_{\mu \nu} \Theta^\mu \Theta^\nu = \eta_{\mu \nu} A^\mu_a(x)dx^a \otimes dx^b = g_{\mu \nu}(x)dx^\mu \otimes dx^\nu\]

\[\Rightarrow \quad g_{ab}(x) = \eta_{\mu \nu} A^\mu_a(x)A^\nu_b(x)\]

\[\Rightarrow \quad \{\Theta^\nu(x)\}\text{ indeed determines }g_{\mu \nu}(x):\]

\[\Rightarrow \quad \text{The } A^\nu_a(x)\text{ carry all physical (here shape) info}!\]
How then does $A^i_j(x)$ encode $\xi^i_k, \omega^i_j$,

1. **Start with orthonormal frame:**

   $$\Theta^i(x) = A^i_j(x) \, dx^j$$

   $$\Theta^i(x) = A^i_j(x) \, dx^j$$

   **Here:**

   $$d\Theta^i(x) = A^i_{jk}(x) \, dx^k \wedge dx^j$$

   Because of \((k)\)

   $$= -\frac{1}{2} C^i_{ab} \Theta^a \wedge \Theta^b = -\frac{1}{2} C^i_{ab} A^a_k A^b_j \, dx^k \wedge dx^j$$

   $$\Rightarrow A^i_{jk} = -\frac{1}{2} C^i_{ab} A^a_k A^b_j$$

   $$\Rightarrow C^i_{ab}(x) = -2 A^i_{jk}(x) \left(A^j(x)\right)^{\prime}_{\prime} \left(A^i(x)\right)^{\prime}_{\prime}$$

2. **The $C^i_{ab}(x)$ yield the $\Gamma^i_{jk}(x)$ through:**

   $$\begin{align*}
   \Gamma^i_{jk}(x) &:= \frac{1}{2} \left[ C^e_{ik} g^{ej} C^s_{sj} - C^s_{jik} g^{ej} C^e_{sji} 
   + \frac{1}{2} g^{ej} \left(g_{ik} g_{sj} + g_{sj} g_{ik} - g_{kj} g_{ij}\right) \right] \\
   \text{(lecture 1)}
   \end{align*}$$

   Notice: This simplifies for orthonormal frames with $g_{ij}(x) = \delta_{ij}$.

3. **The $\Gamma^i_{jk}(x)$ yield the $\omega^i_j(x)$:**

   $$\omega^i_j(x) := \Gamma^i_{jk}(x) \Theta^k(x)$$

4. **Recall the 2nd structure equation:**

   $$\Omega^i_j(x) := d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

   $$\Rightarrow \text{We have: } A^i_j \rightarrow \Theta^i \rightarrow C^i_{jk} \rightarrow \Gamma^i_{jk} \rightarrow \omega^i_j \rightarrow \Omega^i_j$$
Recall important identities: (torsionless case)

- **Structure eqn. I:**
  \[ \Theta^i = D \Theta^i = d \Theta^i + \omega^i_j \wedge \Theta^j = 0 \]
  (Ordinary: \( \Theta^i = dx^i \Rightarrow d \Theta^i = 0 \)
  and \( \omega^i_j \wedge \Theta^j = 0 \iff \Gamma^i_{jk} = \nu^i_{jk} \])

- **Structure eqn. II:**
  \[ \Omega^i_j = d \omega^i_j + \omega^i_k \wedge \omega^k_j \]
  (Recall: \( R^i_{jkl} = \Gamma^i_{jk} \wedge \Gamma^k_{li} + \Gamma^k_{jl} \wedge \Gamma^i_{ki} \))

- **Bianchi identity I:**
  \[ \Omega^i_j \wedge \Theta^j = 0 \]

- **Bianchi identity II:**
  \[ D \Omega^i_j = 0 \]
  (From diffeomorphism invariance)

And, in the case of ON frames:
\[ \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \]

---

**Tetrad formulation of GR:**

Consider the action, for now, without cosmological constant and without matter:

\[ S_{\text{grav}} = \frac{1}{16\pi G} \int_B R \sqrt{g} \, d^4x \]

**Recall Hodge \( \star \):**

\[ \star = \frac{1}{p!} \epsilon_{i_1 \ldots i_p} \Theta^{i_1} \wedge \ldots \wedge \Theta^{i_p} \]

then
\[ \star \epsilon_{i_1 \ldots i_p} \Theta^{i_1} \wedge \ldots \wedge \Theta^{i_p} = \pm 1 \quad \text{totally anti-symmetric} \]

i.e.
\[ \star : \Lambda^p \rightarrow \Lambda^{n-p} \]

**Thus:**

\[ S_{\text{grav}} = \frac{1}{16\pi G} \int_B \star R \]

4-form
Aim now: Re-express \( S'_{\mu \nu} \) in terms of \( \Theta^\rho \) and \( \Omega^{\rho \nu} \).

- **Define:** 
  
  "capital \( \Omega \)" is a \((0, 2)\) tensor-valued 2-form

  \[
  H_{\mu \rho} := * (\Theta^\rho \wedge \Theta^\sigma) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \rho \sigma} \Theta^\sigma \wedge \Theta^\delta
  \]

  \[
  H_{\mu \rho \sigma} := * (\Theta^\rho \wedge \Theta^\sigma \wedge \Theta^\delta) = \frac{1}{2} \sqrt{g} \varepsilon_{\mu \rho \sigma \delta} \Theta^\delta
  \]

  \( \Theta \) is a \((0, 3)\) tensor-valued 1-form.

- **Proposition:**

  \( \star R = H_{\mu \nu} \wedge \Omega^{\mu \nu} \)  
  \( \star \) is a \((0, 2)\) tensor-valued

  i.e.: \( S'_{\mu \nu} (\Theta^\rho) = \int H_{\mu \nu} \wedge \Omega^{\mu \nu} \)

- **Proof:**

  Use \( \Omega^{\rho \nu} = \frac{1}{2} R^{\rho \nu \kappa \lambda} \Theta^\kappa \wedge \Theta^\lambda \Rightarrow \)

  \[
  H_{\mu \nu} \wedge \Omega^{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} R^{\rho \nu \kappa \lambda} \Theta^\kappa \wedge \Theta^\sigma \wedge \Theta^\mu \wedge \Theta^\lambda
  \]

  Use also: \( \varepsilon_{\mu \nu \rho} \varepsilon_{\rho \kappa \lambda} = 2 (\delta_{\mu \kappa} \delta_{\nu \lambda} - \delta_{\mu \lambda} \delta_{\nu \kappa}) \Rightarrow \)

  \[
  H_{\mu \nu} \wedge \Omega^{\mu \nu} = \frac{4}{3} R^{\rho \nu \mu \nu} \Theta^\rho \wedge \Theta^\sigma \wedge \Theta^\delta \wedge \Theta^\chi = \star R \checkmark
  \]

- **Proposition:** \( \nabla H_{\mu \nu} = 0 \)

  Recall the "first standard equation" \( \nabla \Theta = 0 \)

- **Proof:** \( \nabla H_{\mu \nu} = \partial (\frac{1}{\sqrt{g}} \varepsilon_{\mu \nu \rho \sigma} \Theta^\rho \wedge \Theta^\sigma) = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} (\partial \Theta^\rho \wedge \Theta^\sigma + \Theta^\sigma \wedge \partial \Theta^\rho) \)
The main proposition:

Variation of the action with respect to $\delta \Theta^\mu(x)$ yields:

\[ S(\ast R) = (\delta \Theta^\mu) \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + \delta (\text{something}) \]

It implies:

\[ 16\pi G \delta S_{\text{grav}} = \int_B \delta \Theta^\mu \wedge H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} + \int_{\partial B} (\text{something}) \]

\[ \delta S_{\text{matter}} = \int_B \delta \Theta^\mu \wedge (\ast T_\mu) \]

⇒ The equation of motion, i.e., the Einstein equation,

\[ \frac{\delta (S_{\text{grav}} + S_{\text{matter}})}{\delta \Theta^\mu} = 0 \]

becomes:

\[ -\frac{i}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} = 8\pi G \ast T_\mu \]

Exercise: add the cosmological constant.

Remark: The Einstein form $G_\mu := G_{\mu\nu} \Theta^\nu$ obeys

\[ \ast G_\mu = -\frac{i}{2} H_{\mu\nu\sigma} \wedge \Omega^{\nu\sigma} \]

⇒

\[ G_\mu = 8\pi G T_\mu \]
Proof of the main proposition:

\[ S(R) = (\delta \Theta^\nu) \wedge H_{\mu \nu} \wedge \Omega^{\nu \sigma} + d(\text{something}) \]

Indeed:

\[ S(R) = (\delta H_{\mu \nu}) \wedge \Omega^{\nu \sigma} + H_{\mu \nu} \wedge \delta \Omega^{\nu \sigma} \]

Consider the first term:

\[ \delta H_{\mu \nu} = \delta \left( \frac{i}{2} \sqrt{g} \varepsilon_{\rho \sigma \mu \nu \delta} \Theta^\rho \wedge \Theta^\sigma \right) \]

\[ = (\delta \Theta^\rho) \wedge H_{\rho \mu \nu} \]

\[ \Rightarrow \quad S(R) = (\delta \Theta^\rho) \wedge H_{\rho \mu \nu} \wedge \Omega^{\nu \sigma} + H_{\mu \nu} \wedge \delta \Omega^{\nu \sigma} \]

Examine this term:

\[ \delta \Omega^{\nu \sigma} = \delta (d \omega^{\nu \sigma} + \omega^{\nu \sigma} \wedge \omega^{\rho \sigma}) \]

\[ = d \delta \omega^{\nu \sigma} + (\delta \omega^{\rho \sigma}) \wedge \omega^{\rho \sigma} + \omega^{\nu \sigma} \wedge \delta \omega^{\rho \sigma} \]

\[ \Rightarrow \quad H_{\mu \nu} \wedge \delta \Omega^{\nu \sigma} = d(H_{\mu \nu} \wedge \delta \omega^{\nu \sigma}) - (d H_{\mu \nu}) \wedge \delta \omega^{\nu \sigma} \]

\[ + H_{\mu \nu} \wedge \delta \omega^{\rho \sigma} \wedge \omega^{\rho \sigma} + H_{\mu \nu} \wedge \omega^{\rho \sigma} \wedge \delta \omega^{\rho \sigma} \]

\[ = (\delta \omega^{\nu \sigma}) \wedge D H_{\mu \nu} + d(H_{\mu \nu} \wedge \delta \omega^{\nu \sigma}) \]

[recall: \( = 0 \) by Prop. above]

\[ \Rightarrow \quad \text{Indeed:} \]

\[ S(R) = (\delta \Theta^\rho) \wedge H_{\rho \mu \nu} \wedge \Omega^{\nu \sigma} + d(H_{\mu \nu} \wedge \delta \omega^{\nu \sigma}) \]
General Relativity as a "gauge theory"

Recall:

\[
\mathcal{S}_\text{Einst}(\Theta^\nu) = \int H_\mu^\nu \wedge \Omega^\nu = \text{Einstein action}
\]

\[-\frac{1}{2} H_{\mu
u\sigma} \wedge \Omega^\nu = 8\pi G \ast T_\mu = \text{Einstein equation}\]

are now the same in all coordinate systems.

In addition:

They are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:

\[
\Theta^\mu(x) \rightarrow \tilde{\Theta}^\mu(x) = A^\nu_\sigma (x) \Theta^\sigma(x)
\]

The \( A^\nu_\sigma (x) \) are local Lorentz transformations.

Upshot: a We can start with any matter theory that is invariant under global Lorentz transformations and, through general relativity, turn it into a theory that is invariant under local Lorentz transformations.

Therefore:

Derivatives become covariant derivatives.

A new field is introduced: gravity's \( \Gamma \).

\[ \rightarrow \text{This is analogous to the gauge principle of particle physics:} \]

\[ \begin{align*}
&\text{A global symmetry is "gauged" to become local.} \\
&\text{Derivatives become covariant derivatives} \\
&\text{A new field is introduced.}
\end{align*} \]
The gauge principle:

Action for a Dirac field (electrons, quarks etc):

$$S'[\Psi] = \int \bar{\Psi} \left( ig^\mu \partial_\mu - m \right) \Psi \ d^4x$$

It has a global symmetry:

$$\Psi(x) \rightarrow \tilde{\Psi}(x) := e^{i \lambda(x)} \Psi(x) \text{, i.e., } \bar{\Psi}(x) \rightarrow \bar{\tilde{\Psi}}(x) = e^{-i \lambda(x)} \bar{\Psi}(x)$$

$$\Rightarrow S[\Psi] \rightarrow S[\tilde{\Psi}] = S'[\Psi]$$

However, no local symmetry:

$$\Psi(x) \rightarrow \tilde{\Psi}(x) := e^{i \lambda(x)} \Psi(x) \quad \bar{\Psi}(x) \rightarrow \bar{\tilde{\Psi}}(x) = e^{-i \lambda(x)} \bar{\Psi}(x)$$

$$S[\Psi] \rightarrow S[\tilde{\Psi}] \neq S'[\Psi]!$$

Gauge principle: Introduce a new field $A_\mu(x)$ that transforms so as to absorb the extra term:

$$S['\Psi, A] := \int \bar{\Psi}(x) \left( ig^\mu \left( \partial_\mu + i A_\mu(x) \right) - m \right) \Psi(x) \ d^4x$$

"Covariant derivative"

Now under

$$\Psi(x) \rightarrow \tilde{\Psi}(x) := e^{i \lambda(x)} \Psi(x)$$

$$A_\mu(x) \rightarrow \tilde{A}_\mu(x) := A_\mu(x) - i \partial_\mu \lambda(x)$$

the action always:

$$S[\Psi, A] \rightarrow S['\tilde{\Psi}, \tilde{A}]$$

$$= \int \bar{\tilde{\Psi}}(x) e^{-i \lambda(x)} \left( ig^\mu \left( \partial_\mu + i A_\mu - i \partial_\mu \lambda - m \right) \right) e^{i \lambda(x)} \Psi(x) \ d^4x$$

$$= S['\Psi, A]$$
Generalization to Yang-Mills theory

Gauging \( \Phi(x) \rightarrow e^{i\alpha(x)} \Phi(x) \) introduced \( A_\mu(x) \), and \( A_\mu(x) \) turns out to exist: The EM 4-potential. We "derived" the electromagnetic force!

Notice: \( e^{i\alpha(x)} \in U(1) \)

\[ U(1) = \{ G \in \mathbb{C} \mid G^* = G^{-1} \} \]

Now give the Dirac particles an extra index (isospin bundle)

\[ S'[\Phi] = \int \bar{\Psi}_{\alpha} \left( i \gamma^\mu \partial_\mu - m \right) \Psi_\beta \, d^4x \quad (\Sigma \text{ implied}) \]

It's invariant under:

\[ \Psi_\alpha(x) \rightarrow G_{\alpha\beta} \Psi_\beta(x) \quad (anny x) \text{ implied} \]

where \( G \in SU(N) \)

\[ SU(N) = \{ G \in M_n(\mathbb{C}) \mid G^* = G^{-1}, \det(G) = 1 \} \]

Now, we gauge, i.e., require invariance under:

\[ \Psi_\alpha(x) \rightarrow G_{\alpha\beta}(x) \Psi_\beta(x) \quad \text{where} \ G \in SU(N) \]

\[ \Rightarrow \text{Invariance of the action now requires new field } B_\mu(x) : \]

\[ S'[\Phi] = \int \bar{\Psi}_{\alpha} \left( i \gamma^\mu \left( \partial_\mu + i B_\mu(x) T^a \right) - m \right) \Psi_\beta \, d^4x \]

"covariant derivative"

and \( B_\mu(x) \rightarrow \tilde{B}_\mu(x) = B_\mu(x) + \text{complicated} \)

Here: \( T^a \in su(N) \) are Lie algebra basis, i.e. they are generators of infinitesimal \( SU(N) \) transformations.

\[ \square \text{ Upshot:} \begin{align*} 
\diamond \quad N=2 & \quad \text{Weak force (though mixed with } N=1 \text{ ex)} \\
\diamond \quad N=3 & \quad \text{Strong force QCD} 
\end{align*} \]
Recall:
\[ S_{\text{Einstein}} = \int \sum \omega \wedge \Omega^\mu \wedge \wedge_\nu \]

\[ -\frac{1}{2} \nabla_{\mu} \sigma \wedge \Omega^\nu = g_{\mu\nu} \wedge T \]

are the same also with any choice of ON bases in the tangent spaces, i.e., we have a local symmetry under:
\[ \Theta^\alpha(x) \rightarrow \widetilde{\Theta}^\alpha(x) = A^\alpha_\beta(x) \Theta^\beta(x) \]
The \[ A^\alpha_\beta(x) \] are local Lorentz transformations.

Our covariant derivative:
\[ \nabla_{e_\mu} (V^\nu e_\mu) = \left( \frac{\partial}{\partial x^\nu} V^\nu(x) \right) e_\theta + V^\nu(x) \omega^\nu_\theta (e_\mu) e_\theta \]

Do the \[ \omega^\nu_\theta \] indeed generate plays role of \[ A_\mu^\nu \]
in infinitesimal Lorentz transformations? but is now gravity!

<-> Interpretation of the connection in ON frames:

Q: The connection 1-forms \[ \omega^\nu_\theta \] are not, we know, tensor-valued 1-forms. Wherein do they take their values?

A: The connection 1-forms take values in the set of infinitesimal Lorentz transformations

Intuition?

The connection yields the change under infinitesimal parallel transport - and parallel transport preserves the metric, i.e. it preserves the lengths of vectors, i.e. the change can only be an infinitesimal "rotation", i.e. an infinitesimal Lorentz transformation.
Recall:  "Lorentz transformations $N^a$" are linear maps obeying:
\[ \Lambda^a_b \Lambda^b_c = \Lambda^a_c \]

⇒ Infinitesimal Lorentz transformations
\[ N^a_a = \delta^a_a + \xi^a_a \]
with \( (\xi^a_a)^2 = 0 \)

obey:
\[ (\delta^a_a + \xi^a_a) (\delta^b_b + \xi^b_b) \eta_{\mu \nu} = \eta_{\mu \nu} \]

i.e.:
\[ \xi^a_a \eta_{\mu \nu} + \xi^b_b \eta_{\mu \nu} = 0 \]

⇒ Infinitesimal Lorentz transformations "JLT" are given by
\[ \forall N^a_a = \delta^a_a + \xi^a_a \] which obey:
\[ \xi_{ba} + \xi_{ab} = 0 \]

Q:  Are connection 1-forms JLT-valued?

Proposition:
In orthonormal frames, the 1-form $\omega_{\mu \nu}$ obeys
\[ \omega_{\mu \nu} + \omega_{\nu \mu} = 0 \]
i.e. it takes values that are infinitesimal Lorentz transformations.

Proof:
Recall: Absolute exterior derivative: (an anti-derivative)
\[ D_{\alpha \beta} c_{\cd} = dt^a_{\alpha \beta} c_{\cd} + \omega_{\alpha \beta \cd} c_{\cd} \]
\[ \uparrow \text{any tensor-valued} \]
\[ \uparrow \text{diffusion form.} \]

Thus:
\[ 0 = \nabla g_{\mu \nu} = Dg_{\mu \nu} = dg_{\mu \nu} - \omega_{\mu \nu} \wedge g_{\nu i} - \omega_{\mu i} g_{\nu i} \]
i.e. \( 0 = \omega_{\mu \nu} + \omega_{\mu \nu} \)