A singularity theorem:

Assume that:

- $(M, g)$ is a globally hyperbolic spacetime
- The energy-momentum tensor of matter obeys the strong energy condition:

\[
(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) g^{\alpha\beta} g_{\alpha\beta} \geq 0 \quad \forall \text{ timelike } \xi.
\]

- There exists a $C^2$ spacelike Cauchy surface $\Sigma$, on which the trace of the extrinsic curvature, $K$, is bounded from above by a negative constant $C$:

\[
K(p) \leq C < 0 \quad \forall p \in \Sigma.
\]

Then:

No past-directed timelike curve from a spacelike hypersurface $\Sigma$ can have eigentime, i.e., proper length, larger than $\frac{3}{C}$.

i.e.: All past-directed timelike geodesics are incomplete.

\[\Rightarrow\] There is a cosmological singularity in the finite past!
Extrinsic curvature?

Later more on this

The extrinsic curvature of a spacelike hypersurface describes how much curvature there is in between the spacelike hypersurface and the time dimension.

Intuitively: \( \Gamma \) is the rate of the expansion of spacetime, more precisely its negative, the rate of contraction.

Thus: Assuming \( K(p) \leq \Gamma < 0 \) meant that spacetime has a finite minimum expansion rate everywhere on \( \Sigma \).

\( \Rightarrow \) We'll define expansion below in detail.

The strong energy condition?

Recall: a The "weak energy condition":

\[ T_{\mu\nu}v^\mu v^\nu \geq 0 \quad \text{for all timelike } v : g(v,v) < 0 \]

Meaning? For an observer with unit tangent \( v \) the local energy density is: \( T_{\mu\nu}v^\mu v^\nu > 0 \)

b The "dominant energy condition":

\[ T_{\mu\nu}v^\mu v^\nu \geq 0 \quad \text{and } K_{\mu} K^{\mu} \leq 0 \]

\( \text{I.e. } T_{\mu\nu}v^\mu \) is non-space-like.

where \( v \) is any timelike vector and \( K_{\mu} := T_{\mu\nu}v^\nu \)

Meaning? The local energy-momentum flow vector \( K \) may not be conserved but has to be non-space-like: Flow should be into the future and for causality.
The "strong energy condition"

Matter is said to obey the strong energy condition iff:

\[(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) g^{\nu\rho} \geq 0 \text{ for all timelike } g.\]

as we will discuss below

Intuition? Excludes matter that causes accelerated expansion.

Plausible? Yes, obeyed by known matter. (but not by dark energy)

Relationship? Independent of weak and dominant energy condition.

Concretely: For known matter, \( T_{\mu\nu} \) is diagonalizable to obtain:

\[ T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & p \end{pmatrix} \]

The energy conditions then read:

- Weak: \( \rho > 0 \) and \( \rho + p_i > 0 \) for \( i \in \{1,2,3\} \)
- Dominant: \( \rho \geq |p_i| \) for \( i \in \{1,2,3\} \)

Exercise: Show this → Strong: \( \rho + \frac{2}{3} \sum_{i=1}^{3} p_i > 0 \) and \( \rho + p_i > 0 \) for \( i \in \{1,2,3\} \)

Recall: A cosmological constant \( \Lambda \) can be viewed as a contribution to \( T_{\mu\nu} \).

Indeed, there is no long known example of \( \rho = 0 \), if \( w = -1 \) [i.e., \( w \neq -1 \)], i.e., if the spacetime is flat.

Exercise: Show that the strong energy condition is violated in cosmology if \( w < -\frac{1}{3} \), i.e., if the expansion is accelerating: \( \ddot{a}(t) > 0 \).
Essence of point e):

Given, in particular, the strong energy condition, one can show that geodesics meet a divergence of a quantity called expansion, \( \Theta \), in finite proper time. Important notion also e.g. in study of gravitational collapse of stars.

The “expansion,” \( \Theta \):

- Consider a “congruence of timelike geodesics” through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \). If parametrized by proper time, their tangent vector field \( \xi \), namely

\[
\xi := \frac{d}{d\tau} \text{ proper time}
\]

will obey: \( g(\xi, \xi) = -1 \) \( \forall p \).

- Consider now a one-parametric subset family of these geodesics:

\[
y(t, s)
\]

\( t \) parametric of family of neighboring geodesics.

\( s \) a “connecting vector field”

Then, we define the deviation vector:

\[
\eta := \frac{d}{ds}
\]

\( \xi \) a line of constant \( \tau \) value

\( \eta \) a geodesic, i.e., a line of constant \( s \) value
How does $\eta$ change along a geodesic?

$\xi, \varsigma$ are Riemann normal coordinates for a geodesic traveller.

$$\Rightarrow \frac{d}{d\tau} \frac{d}{d\varsigma} = \frac{d}{d\tau} \frac{d}{d\xi}, \text{ i.e., } [\xi, \eta] = 0$$

Since the torsion vanishes:

$$0 = \mathcal{T}(\xi, \eta) = \xi_\eta \eta - \eta_\xi \xi - [\xi, \eta]$$

$$\Rightarrow \xi^\nu \eta = \eta^\nu$$

$$\Rightarrow \xi^\nu \eta_\mu \eta^\nu = \eta^\nu \xi^\nu \eta^\nu$$

$$\Rightarrow \xi^\nu \eta_\mu = \eta^\nu \xi^\nu$$

$$\Rightarrow \xi^\nu \eta_\mu = \eta^\nu \xi^\nu = g_{\nu\mu} \Rightarrow B_{\nu}^{\phantom{\nu} \mu} = \xi^\nu$$

Along the geodesic's direction, $\xi$, the deviation vector $\eta^\nu$ changes its direction and length by $B_{\nu}^{\phantom{\nu} \mu} \eta^\mu$.

The tensor $B_{\nu}^{\phantom{\nu} \mu}$ can be decomposed covariantly and uniquely into:

$$B_{\nu}^{\phantom{\nu} \mu} = \omega_{\mu\nu} + S_{\mu\nu} + T_{\mu\nu}$$

(All 3 terms are tensors because the split is covariant)

We have: $\omega_{\mu\nu} = \frac{1}{2} (B_{\mu\nu} - B_{\nu\mu})$, clearly.

But $S_{\mu\nu}, T_{\mu\nu} = 0$.

In preparation: define the projector $h_{\mu\nu}$ onto $(\mathbb{R}^3)^\perp$; i.e. onto the spatial components:

$$h_{\mu\nu} := g_{\mu\nu} + \xi_\mu \xi_\nu$$

Cheat: is $h_{\mu\nu} w^\nu$ really always $\perp$ to $\xi$?

Indeed: $\xi^\nu h_{\mu\nu} w^\nu = (\xi, w) + (\xi, \xi)(\xi, w) = 0$
Define: The "expansion", $\Theta$, is defined as the magnitude of the spatial part of $B$:

$$\Theta := B_{\mu}^{\nu} h_{\mu\nu}$$

Claim: $Tr(B) = \Theta$

Indeed: $\Theta = B_{\mu}^{\nu} h_{\mu\nu} = B_{\mu}^{\nu} g_{\mu\nu} + \delta^{\nu}_{\omega} \delta_{\omega}^{\mu} B_{\mu}^{\nu}$

$$= Tr(B) + \delta^{\nu}_{\omega} \delta_{\omega}^{\mu} B_{\mu}^{\nu} (= 0 \text{ because } \delta^{\nu}_{\nu} = 0)$$

Therefore:

$$\Theta = \frac{1}{2} (B_{\mu\nu} + B_{\nu\mu}) - \frac{1}{3} \Theta h_{\mu\nu}$$

(because: $Tr(h_{\mu\nu}) = 2 \delta^{\lambda}_{\lambda} h_{\mu\nu} = 2 \delta^{\mu}_{\mu} = 2$)

and:

$$\delta_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu} \Leftarrow \text{the part of } B_{\mu\nu} \text{ which is symmetric and traceless.}$$

Interpretation:

a) $\omega_{\mu\nu}$ is antisymmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\Rightarrow$ it generates Lorentz transformation for $g$

but all $g$ are $\perp$ to the time direction

$\Rightarrow$ $\omega_{\mu\nu}$ generates spatial rotations of neighboring geodesics around another. So, $\omega_{\mu\nu}$ is called

$\omega$ "Twists tensor"

One can prove: (non-trivial)

If one chooses the congruence of geodesics $\Sigma_1$ to $\Sigma_2$, then $\omega_{\mu\nu} = 0$. 
b) $\sigma_{\mu\nu}$ is symmetric, $\sigma_{\mu\nu} = \sigma_{\nu\mu}$. (i.e. hermitian)

Consider “diagonalized”, by suitable choice of co-basis.

$\Rightarrow \sigma_{\mu\nu}$ changes the relative lengths of the basis vectors, by multiplying them with its eigenvalues.

i.e. points on a sphere will under geodesic flow become points on an ellipsoid.

Note: Since $Tr(\sigma) = 0$ we have $det(e^{\nu\sigma}) = 1$

$\Rightarrow$ The volume spanned by basis' vectors stays the same under the action of $\sigma$.

$\Rightarrow \text{Definition: } \sigma_{\mu\nu} = \text{“Shear tensor”}$

c) While the twist and shear tensors are both traceless and therefore volume-preserving, we see that the twist part, $\Theta$, i.e., more precisely $\Theta_{\mu\nu} = \frac{1}{3} \Theta h_{\mu\nu}$.

$\Theta_{\mu\nu} = \text{“Expansion tensor”}$

is indeed generating the spatial expansion or contraction of nearby geodesics!

Evolution of $\Theta$ along a geodesic?
Recall:

Given, in particular, the strong energy condition, our singularity theorem claimed that geodesics must meet a divergence of a quantity called expansion, \( \Theta \), in finite proper time in the past and this will mean a big bang singularity.

The "expansion", \( \Theta \):

- Consider a "congruence of timelike geodesics" through \( \Sigma \), i.e., a smooth family of timelike geodesics, exactly one through each \( p \in \Sigma \) : (\( \Sigma \) is a (causal) surface)

- We consider a one-parametric sub-family of these geodesics:

\[ \gamma(t, s) \]

\( t \) = parameter of family of neighboring geodesics.

\( s \) = parameter of family of geodesics.

\[ \eta := \frac{\partial}{\partial s} \]

- Then, we define the deviation vector to a neighboring geodesic:

\[ \eta \]

- The singularity theorem claims that this happened in the past:

\[ \bigbang \]
How does \( \xi \) change along a post-chorded timelike geodesic with tangent \( \xi \)?

We showed:

\[
\xi^\rho \xi_\rho = \xi^\rho B^\rho_\mu \quad \text{where} \quad B^\rho_\mu := \xi_\rho \xi_\mu
\]

\[\Rightarrow\] Along the geodesic, \( \xi \), the deviation vector \( \xi^\rho \)
changes its direction and length by \( B^\rho_\mu \xi^\mu \).

- The tensor \( B^\rho_\mu \) can be decomposed covariantly
  and uniquely:

\[
B^\rho_\mu = \omega^\rho_\mu + \sigma^\rho_\mu + \tau^\rho_\mu
\]

- Symmetric and trace \( = 0 \)
- Anti-symmetric

Explicitly:

\[
\omega^\rho_\mu = \frac{1}{2} (B^\rho_\mu - B^\mu_\rho)
\]

Twist: \( 0 \rightarrow 0 \)

Volume preserving:

\[
\sigma^\rho_\mu = \frac{1}{2} (B^\rho_\mu + B^\mu_\rho) - \frac{1}{3} \Theta h^\rho_\mu
\]

Shear: \( 0 \rightarrow 0 \)

Volume changing:

\[
\tau^\rho_\mu = \frac{1}{3} \Theta h^\rho_\mu
\]

Expansion: \( 0 \rightarrow \bigcirc \)

Here, we defined: \( \Theta := B^\rho_\mu g^\rho_\mu \) and \( h^\rho_\mu := g^\rho_\mu + \xi^\rho_\mu \xi^\mu \)

i.e., the Expansion, \( \Theta \), is the trace of \( B \), which we showed is
also equal to the magnitude of the spatial part of \( B \): \( \Theta = B^\rho_\mu h^\rho_\mu. \)

Key question: What is the dynamics of \( \Theta ? \)
The Raychaudhuri equation

For the derivation, we will use:

A) Definition of \( B \) is: \( B_{\rho \sigma} := \xi_{\rho \sigma} \)
B) The curvature tensor obeys the Ricci equation:

\[
\xi^a_{\ jbc} - \xi^a_{\ jcb} = R^a_{\ bed} \xi^d
\]

c) \( \xi \) is tangent to a geodesic, i.e., it obeys: \( \nabla \xi = 0 \)

i.e.: \( 0 = \nabla_e \xi^b = \xi^a \nabla_e \xi^b_e = \xi^a \xi^b_{\ ja} e_b \)

True for all \( e_a \), thus: \( \xi^a \xi^b_{\ ja} = 0 \)

Now calculate the rate of change of \( B \) along the geodesic:

\[
\xi^c B_{abc} \quad (A)
\]

\[
\nabla^c B \quad (B)
\]

\[
\equiv \xi^c \xi^d_{\ jbc} + \xi^c R_{abcd} \xi^d
\]

Let rule:

\[
(\xi^c \xi^d_{\ jbc})_{\ b} - \xi^c j_{\ b} \xi^d_{\ jbc} + R_{abcd} \xi^e \xi^d
\]

\[
\equiv - \xi^c j_{\ b} \xi^d_{\ jbc} + R_{abcd} \xi^e \xi^d
\]

\[
\equiv - B^c_{\ b} B_{ac} + R_{abcd} \xi^e \xi^d
\]
In summary, we derived:

\[ \xi^c \mathcal{B}^a_{b e} = -B^c_{b e} B_{a c} + R_{a e d c} \xi^d \xi^c \]  

\[ \text{(**) } \]

The trace of (**) will be the Raychaudhuri equation.

But first, we recall:

\[ \xi^c = \frac{d}{dt} \]

\[ \text{Tr}\ B = B_{\mu \nu} g^{\mu \nu} = \Theta \]

\[ \Rightarrow \text{Trace(LHS) of (**) reads } \frac{d}{dt} \Theta ! \]

Now on the RHS of (**) use the decomposition

\[ B_{\mu \nu} = \omega_{\mu \nu} + \sigma_{\mu \nu} + \frac{1}{3} \Theta h_{\mu \nu} \]  

to express \( B^c_{b e} B_{a c} \):

\[ B^c_{b e} B_{a c} = \omega^c_b (\omega_{a e} + \sigma_{a e} + \frac{1}{3} \Theta h_{a e}) \]

\[ + \sigma^c_b (\omega_{a e} + \sigma_{a e} + \frac{1}{3} \Theta h_{a e}) \]

\[ + \frac{1}{3} \Theta h^c_b (\omega_{a e} + \sigma_{a e} + \frac{1}{3} \Theta h_{a e}) \]

When taking the trace, \( g^{a b} B^c_{b e} B_{a c} \), only the diagonal terms survive:

\[ \text{Tr}(BB) = g^{a b} B^c_{b c} B_{a c} = \omega_{a b} \omega^{a b} + \sigma_{a b} \sigma^{a b} + \frac{1}{3} \Theta^2 h_{a b} h^{a b} \]

The Raychaudhuri equation is then the trace of Eq. (**):

\[ \frac{d\Theta}{dt} = -\frac{1}{3} \Theta^2 - \sigma_{a b} \sigma^{a b} - \omega_{a b} \omega^{a b} - \text{Ricci tensor terms} \]

\[ \text{always positive} \]

\[ \text{total energy} \]

?
Dynamics

Assume that

\[ R_{\mu \nu} \xi^\mu \xi^\nu \geq 0 \] for all timelike \( \xi \)

i.e., using the Einstein equation

\[ R_{\mu \nu} = 8 \pi G \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \]

we are assuming that

\[ T_{\mu \nu} \xi^\mu \xi^\nu - \frac{1}{2} \xi_{\rho} \xi^\rho T \geq 0 \] where \( \xi^{\mu} \xi_{\mu} \leq 0 \)

i.e. the Strong Energy Condition.

Thus, assuming the strong energy condition:

\[ \frac{d \theta}{d \tau} + \frac{1}{3} \theta^2 \leq 0 \]

i.e.,

\[ -\frac{1}{\theta^2} \frac{d \theta}{d \tau} - \frac{1}{3} \geq 0 \]

i.e.,

\[ \frac{d}{d \tau} \theta^{-1} \geq \frac{1}{3} \]

Consider the cases when the geodesics are initially all either

a.) diverging, i.e., \( \theta(\tau_i) > 0 \) (expanding universe) or

b.) converging, i.e., \( \theta(\tau_i) < 0 \) (contracting universe)

(This is reformulating the theorem’s assumption that the extrinsic curvature (i.e. the expansion or contraction at some time exceeds a certain finite value everywhere).
We see that $\theta'(t)$ must have hit $\theta''(t) = 0$ at a finite time $t_{BB}$ (Big Bang).

We see that $\theta'(t)$ will hit $\theta''(t) = 0$ at a finite time $t_{CC}$ (Big Crunch).

**Conclusion:**

Eq. (4) implies that $\theta(t)$ must go through 0, i.e.:

a.) for sufficiently early $t$, have $\theta \rightarrow +\infty$, i.e.: Big Bang

b.) for sufficiently late $t$, have $\theta \rightarrow -\infty$, i.e.: Big Crunch

**Note:** This type of reasoning leads also to further cosmological singularity theorems.

E.g., another cosmological singularity theorem does not assume global hyperbolicity, and its conclusion is weaker:

There is at least one incomplete timelike geodesic.