The "physicist's definition of $T_p(M)$"

Recall: We obtain concrete representations for $p \in M$ and $f \in F(p)$ and $\xi \in T_p(M)$ using a chart $\chi: M \to \mathbb{R}^m$:

Recall: Def's used

pre-composition:

$\chi'^* g \circ \chi = g \circ \chi$ 

$T_p(\chi'^* \xi) \circ \chi = \xi^*$

Terminology: $\xi^*$ is called the "pullback" of $\chi$ 

$T_p \chi^*$ is called the "pullback" of $\chi^*$

Namely:

- Each $p \in M$ has now a concrete image $q \in \mathbb{R}^m$, i.e., it has 'coordinates'.
- Each $f \in F(p)$ is in the image of a concrete function $g \in F(q)$.
- Each $\xi \in T_p(M)$ has now a concrete image $\eta \in T_q(\mathbb{R}^m)$ which we know has the form:

$$\eta = \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i} \bigg|_{x=q}$$

coefficients $\in \mathbb{R}$
**Question:**

Given a $p \in M$ and a $\xi \in T_p(M)$, how do their coordinates and coefficients change under a change of charts?

\[ \phi = \Phi \circ \alpha^{-1}, \text{ i.e.: } \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ \Rightarrow \text{ When changing from chart } \alpha \text{ to chart } \beta : \]

\[ \Phi = \Phi \circ \alpha^{-1} \]
1. Every point \( p \in M \) now has 2 images,
\[
g = (x', ..., x^m) \quad \text{and} \quad s = (\tilde{x}', ..., \tilde{x}^m)
\]
\[
(\tilde{x}', ..., \tilde{x}^m) = \phi(x', ..., x^m)
\]
(concretely: \( \tilde{x}' = \phi'(x', ..., x^m) \)).

2. Every function germ \( f \in \mathfrak{F}_p(M) \) has 2 pre-images,
\[
g \in \mathfrak{F}_q(\mathbb{R}^m) \quad \text{and} \quad h \in \mathfrak{F}_s(\mathbb{R}^m), \text{ related by}
\]
\[
f(p) = g(q) = h(s) \quad ( \in \mathbb{R} ) \quad \text{and by}
\]
\[
h(\tilde{x}', ..., \tilde{x}^m) = g(x', ..., x^m) \quad (\ast) \quad (\text{in a neighborhood})
\]

3. Every tangent vector \( \xi \in T_p(M) \) now has 2 images,
\[
\eta \in T_q(\mathbb{R}^m) \quad \text{and} \quad \nu \in T_s(\mathbb{R}^m).
\]

By construction: \( b/c \) of precomposition:

\[
\eta(g) = \xi(f) = \nu(h) \quad ( \in \mathbb{R} )
\]

\( \Rightarrow \) in particular:

\[
\sum_{i=1}^{m} \eta_i \frac{\partial}{\partial x_i} g(x', ..., x^m) \bigg|_{x=q} = \sum_{j=1}^{m} \nu_j \frac{\partial}{\partial x_j} h(\tilde{x}', ..., \tilde{x}^m) \bigg|_{\tilde{x}=s}
\]

by \( (\ast) \)

\[
= \sum_{j=1}^{m} \nu_j \frac{\partial}{\partial x_j} \left. \frac{\partial}{\partial x^k} g(x', ..., x^m) \right|_{x=q}
\]

\[
= \sum_{j=1}^{m} \nu_j \frac{\partial}{\partial x_j} \left. \frac{\partial}{\partial x^k} g(x', ..., x^m) \right|_{x=q}
\]
Must be true for all $g$!

$$
\sum_{i=1}^{m} \gamma^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \nu^i \frac{\partial}{\partial \psi^i} \bigg|_{x=q} \\
$$

The $\{\frac{\partial}{\partial x^i}\}$ are linearly independent.

$\Downarrow$ Jacobian matrix $D\psi^{-1}$ of $\psi$ at $q$.

$$
\gamma^i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \psi^i} \nu^j \\
$$

$\Downarrow$ Jacobian matrix $D\phi$ of $\phi$ at $q$.

$\Rightarrow$ conversely:

$$
\nu^i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \psi^i} \eta^j \\
$$

Summary:

Given $\xi \in T_p(M)$, its images in charts $\alpha, \beta$, namely $\gamma = \sum_{i=1}^{m} \gamma^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{m} \nu^i \frac{\partial}{\partial \psi^i}$, are related by

$$
\nu^i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \psi^i} \eta^j = \sum_{j=1}^{n} \frac{\partial \phi(x_1, \ldots, x^n)}{\partial x^j} \bigg|_{x=q} \eta^j \\
$$

This transformation property can also be used as the starting point for a definition of tangent vectors!
The "physicist's definition of $T_p(M)$"

**Def**: A tangent vector $\xi \in T_p^\text{(phys)}(M)$ is a map that assigns to each (germ of a) chart a coefficient vector $c \in \mathbb{R}^m$, so that if

- $(\eta^1, \ldots, \eta^m)$ is coefficient vector w. r. e. h. c. chart $\phi$
- $(\eta'^1, \ldots, \eta'^m)$ is coefficient vector w. r. e. chart $\beta$

then:

$$\xi^i = \sum_{j=1}^m \left. \frac{\partial \tilde{x}^i}{\partial x^j} \right|_{x = \phi(p)} \eta^j \quad \text{with} \quad \tilde{x}^i = \phi(x)$$

$$\phi = \beta \circ \alpha^{-1}$$

So far, 2 equiv. defs. of $T_p(M)$:

In a chart, $\phi$, a tangent vector $\xi \in T_p^\text{phys}(M)$ is:

- algebraically:
  $$\xi^i = \sum_{j=1}^m \left. \frac{\partial \tilde{x}^i}{\partial x^j} \right|_{x = \phi(p)} \eta^j$$
  i.e. it is a directional derivative
  
  **Defining property**: Leibniz rule.

- physically:
  $$\xi = (\eta^1, \ldots, \eta^m)$$
  i.e. it is just the direction vector,
  
  **Defining property**: chart change transformation rule

Finally:
The "geometric definition of $T_p(M)$":

**Idea:** Tangent vectors as tangents to paths.

Consider paths in $M$ that pass through $p$:

$\gamma_t : \mathbb{R} \to M$

$\gamma_t(0) = p$

**Note:** For any $f : M \to \mathbb{R}$, we obtain:

$f \circ \gamma_t : \mathbb{R} \to \mathbb{R}$

**Define:**

Two differentiable paths, $\gamma_a$, $\gamma_b$ are called equivalent, if for all $f \in F_p(M)$:

$$\frac{d}{dt} (f \circ \gamma_t) \bigg|_{t=0} = \frac{d}{dt} (f \circ \gamma_b) \bigg|_{t=0}$$

**Intuition:** Two paths $\gamma_a$, $\gamma_b$ are equivalent if they have the same "velocity" at $p$.

**Definition:** $T_p(M)$ is the set of equivalence classes of differentiable paths through $p$.
Are \( T_p(M)^{(\text{geom})} \) and \( T_p(M)^{(\text{alg})} \) equivalent?

Yes!

- Each path \( \gamma \) defines a linear map \( \tilde{\gamma} \):
  \[
  \tilde{\gamma} : f(\gamma) \rightarrow \mathbb{R} \\
  \tilde{\gamma} : f \rightarrow \frac{d}{dt} (f \circ \gamma) \bigg|_{t=0}
  \]

- These \( \tilde{\gamma} \) obey the Leibnitz rule:
  \[
  \tilde{\gamma}(fg) = \frac{d}{dt} (f \circ g)(\gamma(t)) \bigg|_{t=0} = \frac{d}{dt} (f(\gamma(t))g(\gamma(t))) \bigg|_{t=0} \\
  = \frac{d}{dt} (f(\gamma(t))) \bigg|_{t=0} g(\gamma(0)) + f(\gamma(0)) \frac{d}{dt} g(\gamma(t)) \bigg|_{t=0} \\
  = \tilde{\gamma}(f) g + f \tilde{\gamma}(g)
  \]

- \( \Rightarrow \tilde{\gamma} \) is an element of \( T_p(M)^{(\text{alg})} \)

The "Cotangent Space" \( T_p(M)^* \):

Recall:

- Given an \( n \)-dimensional vector space \( V \),
  the set of linear maps \( \omega : V \rightarrow \mathbb{R} \) forms
  also an \( n \)-dim. vector space. It is
  called the "dual space", and denoted \( V^* \).

Definition:

The dual vector space to \( T_p(M) \)

is called the Cotangent Space,

and denoted \( T_p(M)^* \).
We notice:

For every (germ of a) function at \( p \), \( f \in \mathcal{F}(p) \)

one naturally obtains an element

\[ df \in T_p(M)^* \]

called the "differential of \( f \)."

Namely:

\[ df : T_p(M) \to \mathbb{R} \]

is the linear map:

\[ df : \xi \to \xi(f) \]

(Note: thus, we can view \( d \) as a map \( \mathcal{F}(p) \to T_p(M) \). See later...)

Concretely: in a c.d.s., i.e., in a chart,

the abstract \( \xi \in T_p(M) \) and \( f \in \mathcal{F}(p) \)

correspond to some \( \eta \in T_q(\mathbb{R}^n) \) and \( g \in \mathcal{F}(q) \).

Then:

\[ dg : T_q(\mathbb{R}^n) \to \mathbb{R} \]

\[ dg : \eta \to \eta(q) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x_1, \ldots, x^n) \bigg|_{x=q} \]

Recall: Since all \( \eta \in T_q(\mathbb{R}^n) \) take the form \( \eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} \bigg|_{x=q} \)

a basis of \( T_q(\mathbb{R}^n) \) is \( \left\{ \frac{\partial}{\partial x^i} \bigg|_{x=q} \right\}_{i=1}^{n} \).
Question: What is the dual basis in $T_q^*(\mathbb{R}^n)$?

Consider the coordinate functions: $x^k: \mathbb{R}^n \rightarrow \mathbb{R}$.

Their differentials $dx^k \in T_q^*(\mathbb{R}^n)$ obey:

$$dx^k: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$dx^k: \left. \frac{2}{\partial x^i} \right|_{x^i = q} = \delta^k_i$$

The dual basis in $T_q^*(\mathbb{R}^n)$ is given by

$$\left\{ dx^k \right\}_{k=1}^n$$

Thus:

Every element $\omega \in T_q^*(\mathbb{R}^n)$ takes the form:

$$\omega = \sum_{i=1}^n \omega_i \, dx^i$$

and its action on:

$$\omega: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\omega \left( \sum_{i=1}^n \eta^i \frac{2}{\partial x^i} \right) = \sum_{i=1}^n \omega_i \left( \sum_{j=1}^n \eta^j \frac{2}{\partial x^j} \right)$$

$$= \sum_{i=1}^n \omega_i \eta^i$$

$$\Rightarrow \omega \left( \sum_{i=1}^n \eta^i \frac{2}{\partial x^i} \right) = \sum_{i=1}^n \omega_i \eta^i \quad (\text{I})$$
In particular: For arbitrary \( g \in F(q) \), its differential \( dg \in T_q^* (\mathbb{R}^m)^* \) must be of the form:

\[
dg = \sum_{k=1}^m w_k \, dx^k \quad \text{with suitable } w_k \in \mathbb{R}.
\]

How to calculate them?

We know:

\[
dg (\gamma) = \gamma (g) = \sum_{i=1}^m \frac{\partial}{\partial x^i} \gamma (g) \bigg|_{x=q} \tag{(II)}
\]

Compare I, II \( \Rightarrow \) \( w_i = \frac{\partial}{\partial x^i} g(x) \bigg|_{x=q} \)

\[
\Rightarrow \quad dg = \sum_{i=1}^m \left( \frac{\partial}{\partial x^i} g(x) \bigg|_{x=q} \right) dx^i
\]

Exercise: (the "pull back" map)

Assume that \( g \in T_q^* (\mathbb{M})^* \), under two charts \( \alpha, \beta \), as above, corresponds to \( \omega \in T_q^* (\mathbb{R}^m)^* \) and \( \mu \in T_q^* (\mathbb{R}^m)^* \) with:

\[
\omega = \sum_{i=1}^m \omega_i \, dx^i \quad \text{and} \quad \mu = \sum_{i=1}^m \mu_i \, d\tilde{x}^i
\]

Show that \( \mu_i = \sum_{q=1}^m \frac{\partial x^q}{\partial \tilde{x}^i} \bigg|_{\tilde{x}=q} \omega_i \)

Notice that this is the inverse of the Jacobian matrix of \( \beta \circ \alpha^{-1} \) at \( q \),

Remark: The physicist's definition of \( T_p^* (\mathbb{M})^* \) uses this.
Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors.
- Elements of $T_p(M)^*$ are called covariant vectors.
- One often writes symbolically

$$\xi = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i} \bigg|_p$$

for $\xi \in T_p(M)$

$$\omega = \sum_{i=1}^{m} \omega^i dx^i$$

for $\omega \in T_p(M)^*$

even without specifying a particular chart.

We are now ready to define tensors:

**Def:** A tensor, $t$, of rank $(r,s)$ is an element of

$$\underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{r \text{ factors}} \otimes \underbrace{T_p(M)^* \otimes \cdots \otimes T_p(M)^*}_{s \text{ factors}}$$

In a chart:

$$t = \sum_{j_1, \ldots, j_s} \sum_{i_1, \ldots, i_r} t^{i_1, \ldots, i_r, j_1, \ldots, j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$t^{i_1, \ldots, i_r, j_1, \ldots, j_s} = \sum_{j_1', \ldots, j_s'} \frac{\partial x^{j_1}}{\partial x^{j_1'}} \cdots \frac{\partial x^{j_s}}{\partial x^{j_s'}} t^{i_1, \ldots, i_r, j_1', \ldots, j_s'}$$

Thus:

$$\underbrace{T_p(M) = T_p(M)}_{\text{a tangent vector is a tensor of rank } (1,0)} \quad \text{and} \quad \underbrace{T_p(M)^* = T_p(M)}_{\text{a cotangent vector is a tensor of rank } (0,1)}$$
Finally: From local to global!

Def: We call $T(M) = \bigcup_{p \in M} \{p, T_p(M)\}$ the **Tangent bundle**.

Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: $M$ is also called the "Base Space".

Recall that all $T_p(M)$ are $n$-dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$.

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre".

Remark: One obtains other fibre bundles by choosing other standard fibers.

E.g.: a Co-tangent bundle $T^*(M)$

- $(r,s)$-tensor bundle $T^r_s(M)$

- A bundle for isomorphisms (vector bundles) and gauge groups (principal bundles)

Def: The map $\Pi: T(M) \to M$

$\Pi: (p, T_p(M)) \to p$ (i.e.: $\Pi(p) = T_p(M)$)

is called the "Bundle Projection".

Def: A section, \( \sigma \), is a map, $\sigma: M \to T(M)$, which is a continuous right inverse of $\Pi$:

$\Pi(\sigma(x)) = x \ \forall x \in M$ (i.e.: $\Pi \circ \sigma = \text{id}$)
Notice: The graph of a "field" is a section of its fibre bundle.

Recall: The graph of a function $f: A \to B$ is:
$$\{(a, f(a)) \mid a \in A\}$$

Def: A tangent vector field is a map $\xi: \mathcal{M} \to \mathcal{T}\mathcal{M}$.

In a chart: $\xi = \sum_i \xi_i(x) \partial / \partial x^i$.

A cotangent vector field is a map $\omega: \mathcal{M} \to \mathcal{T}^*\mathcal{M}$.

In a chart: $\omega = \sum_i \omega_i(x) dx^i$.

Similarly, tensor fields: $t: \mathcal{M} \to \mathcal{T}^n\mathcal{M}$.

In a chart: $t = \sum_i t_i(x) \partial / \partial x^i \wedge \cdots \wedge \partial / \partial x^i$.

Why then fibre bundles? To capture global nontriviality.

Fibre bundles are required to be locally trivial:

$\mathcal{M}$ can be covered with neighbourhoods $U_i$,

so that $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$.

But fibre bundles are allowed to be globally nontrivial:

For a suitable vector bundle $B$, we can have

$\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$

$\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$.

But in the overlap regions, the two isomorphisms may differ $\Rightarrow B \neq \mathcal{M} \times \mathbb{R}^n$.

(The isomorphisms may differ by elements of $GL_n(\mathbb{R})$, the "structure group" here.)
**Definition:** For the algebra of \( C^\infty \) functions \( M \to \mathbb{R} \), we write \( \mathcal{F}(M) \).

**Note:** One can show that contravariant vector fields are the derivations of the algebra \( \mathcal{F}(M) \), i.e.:

If \( \xi \) is a contravariant vector field, then

\[
\xi : \mathcal{F}(M) \to \mathcal{F}(M)
\]

is linear and obeys the Leibniz rule:

\[
\xi(fg) = \xi(f)g + f \xi(g)
\]

for all \( f, g \in \mathcal{F}(M) \).

**Next topic:** Differential forms:

We already have covered some differential forms:

- The set \( \Lambda_0 := \mathcal{F}(M) \) is called the set of 0-forms.

- The set of covariant vector fields is denoted \( \Lambda_1 \) and called the set of 1-forms.

- For \( r = 2, 3, \ldots \) the set, \( \Lambda_r \), of \( r \)-forms is defined to be the set of totally anti-symmetric tensor fields of rank \((0, r)\).