Differential forms (also called "exterior differential forms")

Why? Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p(M)^*$ at $p$:

- Each $\omega \in T_p(M)^*$ is a linear map:
  
  \[ \omega : T_p(M) \rightarrow \mathbb{R} \]

- $\omega : \xi \mapsto \omega(\xi)$

- Each such $\omega$ is a covariant tensor, of rank $(0, 1)$

More generally, consider the covariant tensors of rank $(0, r)$:

- Recall: $T_p(M)_r := T_p(M)^* \otimes \ldots \otimes T_p(M)^*$

- Each $\nu \in T_p(M)_r$ is a multi-linear map:
  
  \[ \nu : T_p(M)^* \rightarrow \mathbb{R} \]

- In particular, if $\xi_1, \ldots, \xi_r \in T_p(M)$ then:
  
  \[ \nu : \xi_1 \times \ldots \times \xi_r \rightarrow \nu(\xi_1, \ldots, \xi_r) \]
**Definition:** If \( r > 1 \) and \( \omega \in T_r(M) \), then we define the "anti-symmetric part of \( \omega \)" as the image \( \tilde{\omega} = A(\omega) \) of \( \omega \) under the linear antisymmetrization map \( A \):

\[
\tilde{\omega}(\xi_1, \ldots, \xi_r) = A(\omega)(\xi_1, \ldots, \xi_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) \, \omega(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(r)})
\]

*The sign \( \pm 1 \) of the permutation \( \sigma \).*

**Why consider these?**

> They will be key for integration! Only antisymmetric cov. tensors transform under chart change so as to match the Jacobian determinant entering in integrals when changing charts.

**Concretely:**

1. Consider \( \omega := df \otimes dg \), for \( f, g \in \mathcal{F}(M) \)

2. Then \( \omega(\xi, \xi) = df(\xi) \otimes dg(\xi) \) (where \( \xi := \xi_1(f) \xi_2(g) \))

3. Apply \( A \):

\[
\tilde{\omega}(\xi_1, \xi_2) = A(\omega)(\xi_1, \xi_2) = \frac{1}{2} \left( df(\xi_1) \otimes dg(\xi_2) - df(\xi_2) \otimes dg(\xi_1) \right)
\]

4. \( \Rightarrow \) We can also write:

\[
A(df \otimes dg) = \frac{1}{2} (df \otimes dg - dg \otimes df)
\]
**Proposition:** \( A \) is a projector, i.e., it obeys\[ A \circ A = A \]

Check in above example:
\[
A \circ A (d\!\,f \!\,\wedge d\!\,g) = A \left( \frac{1}{2} d\!\,f \!\,\wedge d\!\,g - \frac{1}{2} d\!\,g \!\,\wedge d\!\,f \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{2} d\!\,f \!\,\wedge d\!\,g - \frac{1}{2} d\!\,g \!\,\wedge d\!\,f \right) - \frac{1}{2} \left( \frac{1}{2} d\!\,g \!\,\wedge d\!\,f - \frac{1}{2} d\!\,f \!\,\wedge d\!\,g \right)
\]
\[
= \frac{1}{2} (d\!\,f \!\,\wedge d\!\,g - d\!\,g \!\,\wedge d\!\,f)
\]
\[
= A (d\!\,f \!\,\wedge d\!\,g)
\]

**Definition:**

For \( r > 1 \) we define the space of differential \( r \)-forms (or ‘exterior’ \( r \)-forms) \( \Lambda^r_p(M) \) at \( p \in M \) as the subspace of totally anti-symmetric tensors of rank \( (0, r) \):

\[
\Lambda^r_p(M) := A T^*_p(M)
\]

\( \Rightarrow \) So if \( \nu \in \Lambda^r_p(M) \) then \( A(\nu) = \nu \)
Definition: □ For \( r = 0 \) we define the set of differential 0-forms at \( p \in M \) as:

\[ \Lambda_0(p) := \mathbb{R} \]

(for 0-forms on the entire manifold we will have \( \Lambda := \mathbb{R} \))

□ For \( r = 1 \) we define the set of differential 1-forms (or "Pfaffian forms") at \( p \in M \) through:

\[ \Lambda_1(p) := T_p(M) \]

Strategy now:

Define multiplication \( \rightarrow \) obtain algebra \( \rightarrow \) obtain derivations

The wedge product:

Def: If \( \omega \in \Lambda_0(p) \), \( \nu \in \Lambda_r(p) \), and \( r, s \geq 0 \) then the wedge product \( \wedge \) yields a new differential form:

\[ \wedge : \Lambda_r(p) \times \Lambda_s(p) \rightarrow \Lambda_{r+s}(p) \]

\[ \wedge : (\omega, \nu) \rightarrow \omega \wedge \nu = \frac{(s+r)!}{s!r!} A(\omega \otimes \nu) \]

a normalization factor

Def: For \( c \in \Lambda_0 \), \( \omega \in \Lambda_s \) we have \( c \wedge \omega = c \omega \)

Note: \( dx_i \wedge dx^i = 0 \) \( \forall i \)

Example: For \( dx^i, dx_i \) we obtain:

\[ dx^i \wedge dx_i = (dx^i \otimes dx_i - dx_i \otimes dx^i) \]
Properties of $\wedge$:

- **Bi-linear:**
  \[(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta\]
  \[(a\omega) \wedge \nu = \omega \wedge (a\nu) = a (\omega \wedge \nu) \quad \forall a \in \mathbb{R}\]

- **associative:**
  \[(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)\]

- "Graded" commutative:
  \[
  \begin{align*}
  \text{E.g., for } dx^i \wedge \Lambda_i(p): \\
  \text{\quad } dx^i \wedge dx^i = -dx^i \wedge dx^i
  \end{align*}
  \]
  \[
  \text{since } s = 1:
  \]
  \[
  \omega \wedge \nu = (-1)^{rs} \nu \wedge \omega \quad \text{if } \omega \in \Lambda_r, \nu \in \Lambda_s
  \]

We can use $\wedge$ to build bases of $\Lambda_r(p)$:

*Assume:* $\{\Theta^i\}_{i=1}^n$ is a basis of $\Lambda_1 = T_p(m)^*$

(for example $\Theta^i = dx^i$)

*Then:* $\{\Theta^i \wedge \Theta^i \wedge \ldots \wedge \Theta^i\}_{1 \leq i_1 < i_2 < \ldots < i_r \leq n}$

Excerpt: show this is a basis of $\Lambda_r(p)$ for $r \geq 1$.

Therefore:

\[
\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

Thus, no diff. forms of degree $r > n$!
Example: For $p \in M = \mathbb{R}^3$ we have bases:

$\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$
$\Lambda_2 = \text{span}(dx^1 dx^2, dx^1 dx^3, dx^2 dx^3)$
$\Lambda_3 = \text{span}(dx^1 dx^2 dx^3)$

Definition:

$\Lambda(p) := \bigoplus_{i=0}^{\dim(\Lambda)} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

  $\omega(p) \in \Lambda(p)$

  It is usually also called simply a differential form and denoted $\omega$.

- These fields form the Grassmann algebra of differential forms, $\Lambda(M)$.
**Recall:**

Given an algebra, it is often useful to consider derivations of the algebra, i.e., to consider linear maps that obey the Leibniz rule. (Similar to how we defined tangent vectors as derivations of $\mathcal{T}(M)$.)

**Here:** For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

**Definition:**

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called a derivation of degree $k$, if:

- $\Phi : \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all $s$
- $\Phi : d \wedge \beta \to \Phi(d) \wedge \beta + d \wedge \Phi(\beta)$ for all $d, \beta \in \Lambda(M)$.

**Also:**
Definition:

A linear map $\Phi : \Lambda (M) \to \Lambda (M)$ is called an anti-derivation of degree $k$, if for all $\omega \in \Lambda ^k (M)$, $\beta \in \Lambda (M)$:

$\Phi : \Lambda _s (M) \to \Lambda _{s+k} (M)$ for all $s$ and

$\Phi : \omega \wedge \beta \to \Phi (\omega) \wedge \beta + (-1)^s \omega \wedge \Phi (\beta)$

"Anti-leibniz rule"

Proposition: (as we will show constructively)

Because of the leibniz rule and linearity, any (anti-)derivation $\Phi : \Lambda (M) \to \Lambda (M)$ is already fully determined by its action only on $\Lambda _0 (M)$ and on a basis of $\Lambda _1 (M)$. 
The exterior derivative

The exterior derivative,

\[ d : \Lambda^r(M) \to \Lambda^{r+1}(M) \]

is the anti-derivation of degree \( r = 1 \) which is defined through:

a) \[ d : \Lambda_0^0(M) \to \Lambda_0^1(M) \]
   \[ d : f \to df \]
   \{ action of \( d \) on \( \Lambda_0^0(M) \) \}

b) \[ d : dx^i \to 0 \] for all \( i \), \{ action of \( d \) on a basis of \( \Lambda_1^0(M) \) \}

In a chart:

\[ \text{We had: } d : f(x) \to df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \ dx^i \]

Now we have more generally:

\[ \beta = \sum_{i_1, \ldots, i_s} \beta_{i_1, \ldots, i_s}(x) \ dx^{i_1} \wedge \cdots \wedge dx^{i_s} \in \Lambda_s^r(M) \]

Recall: \( f \wedge w = f w \) when \( f \in C^\infty(M) \) and \( w \in \Lambda \)

Q: So how to carry out \( d : \beta \to d\beta \)?

A: By applying the anti-Leibnitz rule:

\[ d\beta = \sum_{i_1, \ldots, i_s} \Theta \beta_{i_1, \ldots, i_s}(x) \ dx^i_1 \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_s} \]
Proposition: \( \Lambda(M) \rightarrow \Lambda(M) \) obeys: 

\[
d \circ d = 0
\]

Proof: 

\[
d \circ d = \sum_{i_1, \ldots, i_l} \frac{\partial^2 \phi}{\partial x^{i_1} \partial x^{i_l}} \left( \frac{\partial \phi}{\partial x^{i_2}} \right) dx^{i_1} dx^{i_2} \cdots dx^{i_l} \wedge \Lambda dx^{i_1} \wedge \cdots \wedge \Lambda dx^{i_l} 
\]

\[
= 0
\]

\[
\Rightarrow \exists \Sigma
\]

Example:

\[
\square \text{For } M = \mathbb{R}^3 \text{ and } f \in C(M) \text{ we have:}
\]

\[
df = \sum_{i=1}^3 \frac{\partial f}{\partial x^i} dx^i
\]

\[
\square \text{Notice: } \left( \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3} \right) \text{ is the}
\]

\[
\text{"Gradient field } \nabla f \text{ of } f \" 
\]

\[
\square \text{Example: Electric field } E = df \text{ from potential } \phi.
\]
Now assume $\gamma \in \Lambda^1(\mathbb{M})$ is an arbitrary (i.e., not necessarily gradient) covariant vector field:
\[
\gamma = \sum_{i=1}^{3} \gamma^i(x) \, dx^i \in \Lambda^1(\mathbb{R}^3)
\]

Then:
\[
\beta = d\gamma = \sum_{i,j} \frac{\partial \gamma^i(x)}{\partial x^j} \, dx^i \wedge dx^j
\]

\[\text{from } dx^i \wedge dx^i = - dx^i \wedge dx^i\]

\[\text{does not occur because } dx^i \wedge dx^i = 0\]

\[\beta^i = \sum_{i,j} \left( \frac{\partial \gamma^i(x)}{\partial x^j} - \frac{\partial \gamma^j(x)}{\partial x^i} \right) \, dx^i \wedge dx^j\]

\[\beta^i = \left\{
\begin{array}{l}
\left( \frac{\partial \gamma^1}{\partial x^2} - \frac{\partial \gamma^2}{\partial x^1} \right) \, dx^1 \wedge dx^2 \\
\left( \frac{\partial \gamma^1}{\partial x^3} - \frac{\partial \gamma^3}{\partial x^1} \right) \, dx^1 \wedge dx^3 \\
\left( \frac{\partial \gamma^2}{\partial x^3} - \frac{\partial \gamma^3}{\partial x^2} \right) \, dx^2 \wedge dx^3
\end{array}\right.\]

\[\text{Notice: } (\beta_1(x), \beta_2(x), \beta_3(x)) \text{ are the components of the curl } \nabla \times \gamma\]

\[\text{It is called a "pseudo vector field"}\]

\[\text{It is really a } 2 \text{-form field in 3 dim.}\]
Recall:

Gradient vector fields are curl free: $\nabla \times (\nabla f) = 0$

This is a special case of $d \circ d = 0$

because if $\beta = dy$ then:

$d \beta = d^2 y = 0$

Definition:

- A differential form $\omega$ is called **closed** if:

  $d \omega = 0$

  *E.g.*: We saw that $\beta := dy$ is closed. Is this example typical?

- A differential form $\omega$ is called **exact** if there exists a $\varphi$ so that

  $\omega = d \varphi$ (\( \varphi \) is like an anti-derivative)
How are closedness and exactness related?

This actually depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

**Simplest case:** Assume $M$ is contractible

- i.e., $\exists \bar{F} : [0,1] \times M \to M$
  - so that $\bar{F}(0,x) = x \quad \forall x$
  - $\bar{F}(1,x) = x_0 \quad \forall x$
  - for some fixed $x_0 \in M$

**Poincaré lemma:**

On any contractible manifold:

- $\gamma$ exact $\iff$ $\gamma$ closed

E.g.

- $\mathbb{R}^n$ is contractible
- $\mathbb{R}^n \setminus \mathbb{S}^2$ is not contractible
  - for some arbitrary point $x \in \mathbb{M}$
In general: We only have
\[ \text{exact } \Rightarrow \text{ closed} \]
which is because \( d^2 = 0 \).

\[ \Rightarrow \] We obtain a tool for classifying the "global topology" of differentiable manifolds (checking for holes, bundles etc.)

How? Take a differentiable manifold and calculate the vector space of closed but not exact differential forms.

A look at the bigger picture:

This method is a special case of a cohomology theory, called "De Rham Cohomology".

Q: What is a cohomology theory? (roughly)
A: A cohomology theory is a map \( C \)

\[ C: \left\{ \text{differentiable manifolds} \right\} \rightarrow \left\{ \text{abelian groups} \right\} \]

which is such that if:

\[ C(M) \not\cong C(N) \Rightarrow M \not\cong N \]

\( \uparrow \) i.e. not isomorphic as abelian groups

\( \uparrow \) i.e. no diffeomorphism exists
Why are cohomology theories useful?

It is much easier to check if two abelian groups are isomorphic than to check if two differentiable manifolds are diffeomorphic.

What are the abelian groups in the case of de Rham cohomology?

They are the vector spaces of closed but not exact differential forms.

A look at the even bigger picture

Comment: All cohomology theories are:

"Natural Transformations" between two "Categories."

Def: A category is a set of objects and morphisms:

\[
\begin{align*}
\exists & A \rightarrow B \text{ and } B \rightarrow C \Rightarrow \exists A \rightarrow C \\
\text{Axioms:} & \\
\text{Associativity:} &
\end{align*}
\]
Examples:

- **Category of vector spaces** \( \text{Vec} \):
  - Objects: all vector spaces
  - Morphisms: linear transformations

- **Category of associative algebras** \( \text{Alg} \):
  - Objects: all assoc. algebras
  - Morphisms: algebra homomorphisms

But also:

- **Category of categories** \( \text{Cat} \):
  - Objects: all categories
  - Morphisms: natural transformations, also called functors.

**A cohomology theory is a morphism between two objects in** \( \text{Cat} \), namely **Diff** and **Abe**:

- \( \text{Diff} \): diffeable maps
- \( \text{Abe} \): abelian groups

Crucial: \( \text{group}(N) \) not homomorphic to \( \text{group}(M) \) \( \Rightarrow M \) not diffeomorphic to \( N \)

Note: This is what category theory was originally developed for.
Example: K-theory

- This cohomology theory uses the fact that:
  - There is only one way to put a vector bundle on a contractible manifold but others can have many non-isomorphic vector bundles.

- Recall that, e.g., for a suitable vector bundle $B$:
  $$\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^m$$
  $$\pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^m$$
  But: $B \not\cong M \times \mathbb{R}^m$

Q: What are the abelian groups here?
A: The vector bundles form an abelian group through Whitney's generalization of direct sum $\oplus$.

In this course: We'll focus on the local properties of the manifolds, such as curvature.

We mention cohomology theory issues only on the side in this course.

Next: The inner derivation of $\Lambda(M)$. 