Recall: The set $\Lambda(M)$ of differential forms on $M$ is an associative algebra, called the Grassmann algebra over $M$.

- The multiplication in $\Lambda(M)$ is the wedge product: $\wedge : \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$

- The exterior derivative $d : \Lambda(M) \rightarrow \Lambda(M)$ is an anti-derivation of degree $K = 1$ of the Grassmann algebra $\Lambda(M)$.

But: How to obtain a directional derivative on $\Lambda(M)$?

Recall: Tangent vectors $\xi$ are directional derivatives on $\Lambda_0(M)$.

Plan now:

A. Define an anti-derivation $i_\xi$ of degree $K = -1$: the inner derivation.

$\xi \text{ will generalise finding a tangent vector to a 1-form to feeding it to a p-form}$

B. Combine $d$, $i_\xi$ to obtain a derivation of degree $K = 0$: the Lie derivative

(And the Lie derivative is going to be the directional derivative for differential forms and tensors)
A. The "Inner Derivation":

- Assume \( \xi \) is a tangent vector field.
- Our aim: to define an anti-derivation, \( i_\xi \), of degree \( K = -1 \), i.e., a linear map
  \[
i_\xi : \Lambda^s(M) \to \Lambda^{s-1}(M)
  \]
  \[
i_\xi : \omega \to i_\xi(\omega)
  \]
  which obeys the anti-Leibniz rule:
  \[
i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^s \omega \wedge i_\xi(\nu)
  \]
  if \( \omega \in \Lambda^r(M) \).

- Definition:
  \[
i_\xi : \Lambda^0 \to 0
  \]
  \[
i_\xi : \Lambda^1 \to \Lambda^0
  \]
  \[
i_\xi : \omega \to \omega(\xi)
  \]

- Recall: By linearity and the anti-Leibniz rule this already defines \( i_\xi : \Lambda(M) \to \Lambda(M) \).

- Proposition: If \( \xi \in \Lambda^s(M) \) then \( i_\xi(\xi) \in \Lambda^{s-1}(M) \)
  maps (\( s-1 \)) tangent vectors \( \xi, \ldots, \xi_{s-1} \) this way:
  \[
i_\xi(\xi)(\xi, \ldots, \xi_{s-1}) := i_\xi(\xi, \xi, \ldots, \xi_{s-1})
  \]
Example: Consider \( \mathcal{J}_0 \colon \mathcal{W} \to \mathcal{W} \)

* What is \( i_\mathcal{J}(\mathcal{J}_0) \in \mathcal{W}(M) ? \) Leibnitz rule:

\[
i_\mathcal{J}(\mathcal{J}_0) = i_\mathcal{J}(\mathcal{J}_0(\mathcal{J}_0)) + i_\mathcal{J}(\mathcal{J}_0(\mathcal{J}_0(\mathcal{J}_0))) = \omega(\mathcal{J}_0(\mathcal{J}_0))
\]

\[
= \omega(\mathcal{J}_0(\mathcal{J}_0)) + \omega(\mathcal{J}_0(\mathcal{J}_0(\mathcal{J}_0)))
\]

* Apply \( i_\mathcal{J}(\mathcal{J}_0) \in \mathcal{W}(M) \) to a tangent vector \( \mathbf{v} \):

\[
i_\mathcal{J}(\mathcal{J}_0)(\mathbf{v}) = \omega(\mathcal{J}_0(\mathcal{J}_0(\mathbf{v})))
\]

* Compare with claim of proposition:

\[
i_\mathcal{J}(\mathcal{J}_0)(\mathbf{v}) = i_\mathcal{J}(\mathcal{J}_0(\mathcal{J}_0(\mathbf{v}))) = i_\mathcal{J}(\mathcal{J}_0(\mathcal{J}_0(\mathbf{v}))) = \omega(\mathcal{J}_0(\mathcal{J}_0(\mathbf{v})))
\]

Recall: \( \omega \wedge \nu = \omega \wedge \nu \wedge \omega \)

Properties of \( i_\mathcal{J} \):

\[ i_\mathcal{J}, i_\mathcal{J}_2 = -i_\mathcal{J}_2 \circ i_\mathcal{J}, \]

Thus, in particular:

\[ i_\mathcal{J} \circ i_\mathcal{J} = 0 \]

Recall: We also have \( d\mathcal{J} d\mathcal{J} = 0 \)

Recall: For \( \xi \in T_p(M), \mathcal{J}_0 \in T_p^*(M) \), we have \( i_\mathcal{J}(\xi) = \xi(\mathcal{J}_0) = \xi(\mathcal{J}_0) \)

Definition: The inner derivation, \( i_\mathcal{J}(\xi) \), of a \( \xi \in N(M) \) is also called the interior product of \( \xi \) and \( \mathcal{J}_0 \).
B. The Lie derivative, $L_\xi$: (algebraic definition)

Vectors $\xi : \Lambda^0(M) \to \Lambda^0(M)$ are directional derivatives.

How to generalize the notion of directional derivative to all of $\Lambda^k(M)$?

We have:
- $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ generalizes the notion of differential $d : \Lambda^0 \to \Lambda^1$, $d : f \mapsto df$ to all of $\Lambda^k(M)$.
- $\iota : \Lambda^k(M) \to \Lambda^k+1(M)$ generalizes the notion of evaluation of vectors $\xi$ on covectors $\omega \in \Lambda^k(M)$ to all of $\Lambda^k(M)$.

**Spoiler:** It will be: $L_\xi = d \circ \iota + \iota \circ d$

To construct $L_\xi$, let us first collect desired properties:

- As a directional derivative, it should be a derivation, not an anti-derivation, i.e.:
  
  $$L_\xi(\omega \wedge \nu) = L_\xi(\omega) \wedge \nu + \omega \wedge L_\xi(\nu)$$

  (Recall that the directional derivative on functions $f(x)$, namely the tangent vectors, are mapping $\Lambda^0(M) \to \Lambda^0(M)$)

- $L_\xi$ should map $r$-forms into $r$-forms:
  $$L_\xi : \Lambda^r(M) \to \Lambda^r(M)$$

  i.e. it should be of degree $K = 0$. In particular:
On functions \( f \in \mathcal{F}(M) = \Lambda^0(M) \) it should be the usual directional derivative:

\[
L_\xi : \Lambda^0(M) \to \Lambda^0(M) \\
L_\xi : f \to \xi(f) \quad (i = \sum_{i=1}^n \xi_i(x) \frac{\partial}{\partial x_i}(x))
\]

Recall: once we define \( L_\xi \) on \( \Lambda^0 \) and a basis of \( \Lambda^1(M) \), then by linearity and the Leibniz rule, \( L_\xi \) will automatically be defined on all of \( \Lambda(M) \).

Consider, therefore, any \( \xi f \in \Lambda^1(M) \), e.g., the basis vectors \( \xi_i \) recall that \( \xi f \) is the gradient vector field of the function \( f \).

Then it is natural to define the directional derivative of a gradient field of a function to be the gradient of the directional derivative of the function: (because derivatives ought to commute and the gradient is a derivative too,

\[
L_\xi : \Lambda^1(M) \to \Lambda^1(M) \\
L_\xi : \xi f \to d(\xi(f)) \quad (\xi \in \Lambda^1(M)) \in \Lambda^1(M)
\]

i.e.:

\[
L_\xi(df) = d(\xi(f)) \quad (D)
\]

directional derivative of gradient = gradient of directional derivative
Question: Now that $L_g$ is a fully defined derivation

$L_g : \Lambda(M) \to \Lambda(M),$

can we relate it to $d$ and $i_g$? Yes:

**Cartan's equation:**

$$L_g = d \circ i_g + i_g \circ d$$

**Proof:**

\[ L_g f = d \circ i_g(f) + i_g(df) = 0 + df(\xi) = g(f) \]

\[ df(\xi) = g(f) \text{ because } d = 0 \]

\[ L_g df = d \circ i_g(df) + i_g \circ df = d(g(f)) \]

J.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative

**Definition:**

For any linear maps $A : \Lambda(M) \to \Lambda(M), B : \Lambda(M) \to \Lambda(M)$ we define their commutator (or Lie-, or Poisson bracket):

$$[A, B] := A \circ B - B \circ A$$

**Examples of maps:**

$$d : \Lambda(M) \to \Lambda(M)$$

$$i_g : \Lambda(M) \to \Lambda(M)$$

$$L_g : \Lambda(M) \to \Lambda(M)$$

For the commutators of $d, i_g$ and $L_g$ one can prove:
Proposition: 
\[ [L_\xi, d] = 0 \]
\[ [L_\xi, L_\eta] = L_{[\xi, \eta]} \]
\[ [L_\xi, i_\eta] = i_{[\xi, \eta]} \]

Exercise: prove this.

Here we used on the right hand side that also vector fields
\[ \xi : \Lambda_0(M) \to \Lambda_0(M), \]
have commutators:
\[ [\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1} \xi_i \eta_j \frac{\partial^2}{\partial x_i \partial x_j} f \]
\[ = \sum_{i,j=1} (\xi_i \frac{\partial}{\partial x_j} - \eta_j \frac{\partial}{\partial x_i}) \frac{\partial^2}{\partial x_i \partial x_j} f \]
\[ = \sum_{i,j=1} \xi_i \eta_j \frac{\partial^2}{\partial x_i \partial x_j} f = \nabla(\xi(\eta(f))) \]

Questions:

Since \( L_\xi \) is the directional derivative on \( \Lambda(M) \):

- Can \( L_\xi \) be extended to a directional derivative for all tensor fields? Yes!

- Can \( L_\xi \) be expressed as a Newton-Leibniz limit similar to

\[ f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \] ? Yes!
The geometric definition of $L_\gamma$:

- Recall that for any path
  \[ y : \mathbb{R} \to \mathcal{M} \]
  \[ y : t \to y(t) \]
  we have a tangent vector \( \gamma' (t) \in T_{y(t)} (\mathcal{M}) \) at each point \( y(t) \) of the path:
  \[ \gamma' (t) : f \to \gamma' (t)(f) = \frac{d}{dt} f(y(t)) \bigg|_{t=t_0} \]
  (the geometric definition of the tangent space)

Definition: For a given vector field, $\mathbf{\xi}$, a path $y$ is called an integral curve of $\mathbf{\xi}$, if

\[ \gamma'(t) = \mathbf{\xi}(y(t)) \]

From theory of ODEs:

For every $p \in \mathcal{M}$ there exists a maximal (i.e. inextendible) $C^1$ integral curve through $p$.

Thus, $\mathbf{\xi}$ yields a "flow" (at least for small $t$, locally):
for a fixed $t$: 

\[ \mathcal{M} \]

i.e., for any fixed value of the flow parameter $t$, each point of $\mathcal{M}$ is mapped into another point of $\mathcal{M}$.

- The flow is a diffeomorphism $\phi_t : \mathcal{M} \to \mathcal{M}$:

\[ \mathcal{M} \xrightarrow{\phi_t} \mathcal{M} \]

- As always, a diffeomorphism of manifolds induces corresponding isomorphisms of the tangent, cotangent, and all tensor spaces at $p$ and at $\phi_t(p)$ respectively:

\[ \phi_t^* : T_p(\mathcal{M})^* \to T_{\phi_t(p)}(\mathcal{M})^* \]

- Recall: A tensor field $\tau$ assigns to each $p \in \mathcal{M}$ a tensor $\tau(p) \in T_p(\mathcal{M})^*$.

**Definition:**

We say that a tensor field $\tau$ is invariant under the flow induced by the vector field $\xi$ if:

\[ \phi_t^* (\tau(p)) = \tau(\phi_t(p)) \quad \forall \, t \quad \forall \, p \]

(The flow produces an image of $\mathcal{M}$ in $\mathcal{M}$; image of the tensor field's value at $p$; tensor field's value at the image of $p$)
Definition:

The Lie derivative of any tensor field \( \tau \) at the point \( p = \gamma(s) \in M \) with respect to the flow induced by a vector field \( \xi \) is defined through:

\[
\mathcal{L}_\xi \tau := \lim_{t \to 0} \frac{1}{t} (\phi^1_t(p)(\xi) - \tau)
\]

i.e.,

\[
\mathcal{L}_\xi \tau(p) = \lim_{t \to 0} \frac{1}{t} \left[ (\phi^1_t(p)(\gamma(t))) - \tau(p) \right]
\]

Explicitly, in a chart:

- \( \phi : \mathbb{R} \to \mathcal{M} \) with infinitesimal flow: \( \dot{x}^i(x) = x^i + t \xi^i(x) + O(t^2) \)

- Jacobian matrix:
  \[
  \frac{\partial \dot{x}^i}{\partial x^j} = \delta^i_j + t \frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)
  \]

- Inverse Jacobian:
  \[
  \frac{\partial x^j}{\partial \dot{x}^i} = \delta^j_i - t \frac{\partial \xi^i(x)}{\partial \dot{x}^j} + O(t^2)
  \]

- Image of tensor at \( \tau(x)_{\xi, \eta} \) under flow, back under \( \tilde{x} \to x \), has the components:

  \[
  \phi^i(x)_{\xi, \eta} = \partial x^j(x) \partial \xi^i(x) \partial \eta^j(x)
  \]

  \[
  = \partial x^j(x + t \xi) (\delta^i_j + t \xi^i_j) \cdots (\delta^i_n + t \xi^i_n) (\delta^\eta_j + t \xi^\eta_j) \cdots (\delta^\eta_n + t \xi^\eta_n) + O(t^2)
  \]
\[ f_{ik} := \frac{2}{\partial x_i} f \]

\[ = \tau^{\bar{i}, \bar{\vdots}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\alpha} \bar{\bar{\alpha}}}(x) + t \tau_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\iota}}(x) \xi^\kappa(x) \]

\[ - t \tau_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\iota}}(x) \xi^\kappa(x) - \cdots - t \tau_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\iota}}(x) \xi^\kappa(x) \]

\[ + t \tau_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\iota}}(x) \xi^\kappa(x) + \cdots + t \tau_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\iota}}(x) \xi^\kappa(x) \]

\[ \Rightarrow \left( L_\xi \tau \right)^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\alpha} \bar{\bar{\alpha}}}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \phi^{\bar{-1}}(\varepsilon \xi), \bar{\alpha} \bar{\iota} \bar{\alpha} \bar{\bar{\alpha}}(x) \right) \]

\[ = \tau^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\kappa}}(x) \xi^\kappa(x) - \tau^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\kappa}}(x) \xi^\kappa(x) - \cdots - \tau^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\kappa}}(x) \xi^\kappa(x) \]

\[ + \tau^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\kappa}}(x) \xi^\kappa(x) + \cdots + \tau^{\bar{i}, \bar{\iota}}_{\bar{\alpha} \bar{\iota} \bar{\kappa} \bar{\iota}, \bar{\alpha} \bar{\kappa}}(x) \xi^\kappa(x) \]

\[ \text{□ Equivalent to algebraic definition of } L_\xi \text{?} \]

\[ \text{Yes: Check, e.g., that action on } \Lambda^0(M) \text{ and } \Lambda^1(M) \text{ is the same:} \]

\[ \text{□ For } \tau \in \Lambda^0(M) \text{ we have } \tau = \tau(x) \]

\[ L_\xi \tau(x) = \xi^\kappa \tau_{\bar{\kappa} \bar{\alpha}} = \xi^\kappa \frac{2}{\partial x \kappa} \tau(x) \text{ is gradient} \checkmark \]

\[ \text{□ } \omega \text{-Vector field: } \tau = \tau_{\bar{\iota}}(x) d x^{\bar{\iota}} \in \Lambda^1(M) \]

\[ L_\xi \tau(x) = \left( \xi^\kappa(x) \tau_{\bar{\iota} \bar{\kappa}}(x) + \tau_{\bar{\kappa} \bar{\iota}}(x) \xi^\kappa(x) \right) d x^{\bar{\iota}} \]

\[ \text{Exercise: verify that this agrees with the algebraically defined action of } L_\xi \text{ on } \Lambda^1(M). \]
Collected properties: (without proof)

1. $L_\xi : T_p(M)^*_\mathbb{R} \to T_p(M)^*_\mathbb{R}$ (i.e. $\omega \rightarrow \xi \lrcorner \omega$)

2. In particular, the Lie derivative of a vector field $\eta$ is:

   $L_\xi : \eta \rightarrow L_\xi (\eta) = [\xi, \eta]$

3. One also finds:

   $L_{\xi + \eta} = L_\xi + L_\eta$

   $L_{[\xi, \eta]} = [L_\xi, L_\eta] = L_\xi \otimes L_\eta - L_\eta \otimes L_\xi$

4. Does it still obey a Leibniz rule?

   Yes: $L_\xi (\tau \otimes \sigma) = L_\xi (\tau) \otimes \sigma + \tau \otimes L_\xi (\sigma)$

   (Tensors form an algebra w. respect to multiplication $\otimes$)