

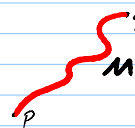
Integration

Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

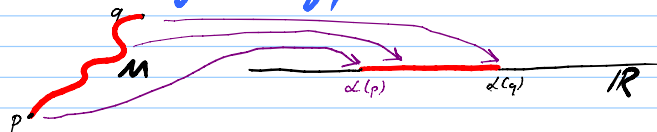
A: Antisymmetry \Rightarrow special transformation property under chart changes:
 $\sim \det(\text{Jacobian})$
 \Rightarrow suitable for integration:
 s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

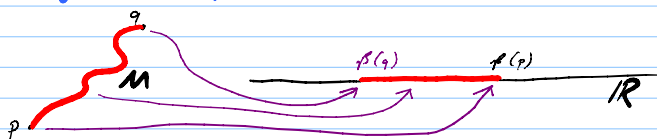
Thus: Before defining integration on mfd's, must study notion of "Orientation" of the mfd.

Namely: Consider e.g. 1-dim mfd: 

□ could have charts of the type



□ or charts of the type



□ But, since $\int_a^b f(t) dt = - \int_b^a f(t) dt'$ one needs to decide!
 because $\frac{dt}{dt'} = -1$ (which is $\det[\text{Jacobian}]$)

For n -dim mflds, may need several charts.

Definitions:

- ▢ A complete collection of charts, i.e., an **Atlas**, A , is called **oriented** if among all overlapping charts with coordinates say x, \tilde{x} the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

- ▢ A mfld M is called **orientable** if it possesses an oriented atlas.

Example: Möbius strips



are not orientable.

- ▢ A mfld, M , together with a choice of oriented atlas, A , is called an **oriented manifold**.
- ▢ Then, an arbitrary chart is called **positive (or negative)** if its Jacobian determinant with charts of the atlas A is positive (or negative).

Definition:

An n -form $\Omega \in \Lambda_n(M)$ is called a volume form if it nowhere

vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:

M possesses a volume form



M is orientable

Integration:

□ Recall change of cds in integration in \mathbb{R}^n :

For $(x^1, \dots, x^n) \rightarrow (\tilde{x}^1, \dots, \tilde{x}^n)$:

Riemann or Lebesgue integrals \rightarrow

$$\int_{\mathbb{R}^n} g(x^1, \dots, x^n) dx^1 \dots dx^n \stackrel{(*)}{=} \int_{\mathbb{R}^n} g(x(\tilde{x})) \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^n$$

\uparrow Jacobian determinant is negative if coordinate systems change handedness.

□ Now for a general n -dimensional diffable mfd M ,

consider an n -form ω in a chart:

$$\omega = f(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x})) \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^2} \dots \frac{\partial x^m}{\partial \tilde{x}^m} \underbrace{d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^m}_{\text{totally antisymmetric!}}$$

○ terms are nonzero only if contain each number $1, \dots, m$ exactly once, e.g. $d\tilde{x}^1 \wedge d\tilde{x}^3 \wedge d\tilde{x}^2 \wedge d\tilde{x}^4 \wedge d\tilde{x}^5 \wedge \dots \wedge d\tilde{x}^m$.

○ Reorder those terms - they are all

$$d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^m$$

up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$$\Rightarrow \omega = f(x(\tilde{x})) \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^m$$

Compare with equation (*) above \Rightarrow

The following definition of the integral of n -forms in an n -dim. diffable mfd is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n -dim mfd

and $\omega \in \Lambda_n(M)$ reads in a chart d : $\omega = f(x) dx^1 \wedge \dots \wedge dx^m$.

Then, if one chart suffices:

$$\int_M \omega := \int_{d(M)} \underbrace{f(x) dx^1 dx^2 \dots dx^m}_{\text{usual Riemann or Lebesgue integral}}$$

$d(M) \leftarrow \text{image of } M \text{ in } \mathbb{R}^m$

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equipped with. That's why orientation is important.

Definition: The boundary operator, ∂

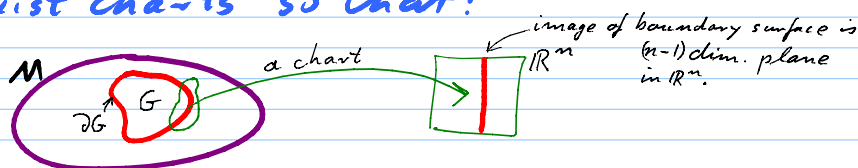
- ▮ Assume $G \subset M$ is a region (i.e. an n -dim., open and connected subset) of the n -dim manifold M .

We denote the $(n-1)$ dim. boundary manifold of G by ∂G :

← the boundary operator

$$\partial G := \text{boundary}(G)$$

- ▮ We say that ∂G is smooth if locally there exist charts so that:



Proposition: If M is orientable, then so is G . Also, the orientation of G induces an orientation of ∂G .

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure \bar{G} of G is a compact n -dim region, then:

$$\int_G d\omega = \int_{\partial G} \omega \quad \text{for all } \omega \in \Lambda_{n-1}(M)$$

Definition: d is also called "co-boundary operator".

Remark:


- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \underbrace{ddw}_{=0 \text{ always for algebraic reasons}} \stackrel{\text{Stokes}}{=} \int_{\partial H} dw \stackrel{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}_{=0}} w$$

= 0 always
for algebraic
reasons.

for geometric reasons
because, indeed,
boundaries don't
possess boundaries:

i.e.: Stokes implies $d^2 = 0 \Leftrightarrow \partial^2 = 0$

E.g.  $G = \text{half sphere}$
 $\partial G = \text{equator}$
 $\partial\partial G = \emptyset$

- Stokes links homology (geometric) to cohomology (algebraic).

Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

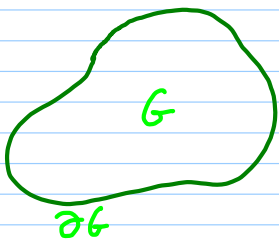
Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f \Big|_a^b \quad (\text{fund. thm of calculus})$$

$$\uparrow = \frac{df}{dx} dx$$

Special case II: "Green's theorem".

- $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G .



↑ recall: this is automatic because $\partial\partial = 0$

- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x) dx^1 + \omega_2(x) dx^2$$

$$\begin{aligned} \text{Then: } d\omega &= d\omega_1(x) \wedge dx^1 + d\omega_2(x) \wedge dx^2 \\ &= \left(\frac{\partial \omega_1}{\partial x^1} dx^1 + \frac{\partial \omega_1}{\partial x^2} dx^2 \right) \wedge dx^1 \\ &\quad + \left(\frac{\partial \omega_2}{\partial x^1} dx^1 + \frac{\partial \omega_2}{\partial x^2} dx^2 \right) \wedge dx^2 \\ &= \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2 + \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 \end{aligned}$$

$$\implies d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 \wedge dx^2$$

Now, Stokes' theorem $\int_G d\omega = \int_{\partial G} \omega$ becomes:

$$\int_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 dx^2 = \int_{\partial G} (\omega_1 dx^1 + \omega_2 dx^2)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G , i.e., a diffeable map, invertible map $\partial G \rightarrow \mathbb{R}$.
- Its inverse is a path: $\gamma: J \subset \mathbb{R} \rightarrow \partial G$, with $\gamma(t) = (x^1(t), x^2(t))$
- Now use $dx^i = \frac{dx^i}{dt} dt$ to obtain an integral over $J \subset \mathbb{R}$

Special case of Green's theorem:

Assume $w \in \Lambda_1$ is closed, i.e., $dw = 0$, i.e., $\frac{\partial w_1}{\partial x^2} - \frac{\partial w_2}{\partial x^1} = 0$

$$\text{Then: } \int_{\partial G} w = 0$$

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

$$\int_{\partial G} w(z) dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff. form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

$$\int_G \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \vec{\nabla} \times \vec{w} \, dG = \int_{\partial G} \vec{w} \cdot d\vec{s}$$

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\vec{\nabla} \times \vec{w}$: vector field

G : a 2dim submanifold of M

∂G : 1dim boundary of A .

is indeed this special case:

$$w \in \Lambda_1(G) \text{ with } \vec{\nabla} \times \vec{w} = dw \in \Lambda_2(G)$$

Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diffeable manifold M ?

□ In \mathbb{R}^n , we had:
$$V = \int_G dx^1 \dots dx^n$$

□ In general, we need to choose a Volume form

$$\Omega \in \Lambda_n$$

obeying $\Omega(p) \neq 0 \forall p \in G$. Then the (Ω -dependent) volume is defined as:

$$V := \int_G \Omega$$

(We will later use the metric tensor to define a volume form for spacetime.)

Proposition: G orientable $\iff \exists$ volume forms Ω
(In fact ∞ many)

Special case IV: Gauss' theorem

To obtain Gauss' theorem we need to define yet a new derivative the divergence of a vector field.

Recall: On \mathbb{R}^n , the divergence of a vector field, ξ , was defined as

$$\text{div } \xi = \sum_{i=1}^n \frac{\partial}{\partial x^i} \xi^i = \xi^i_{,i}$$

\implies How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i_{,i}$ before?

Recall: $(L_{\xi} \tau)_{j_1 \dots j_s}^{i_1 \dots i_s}(x) = \tau_{j_1 \dots j_s, k}^{i_1 \dots i_s}(x) \xi^k(x) - \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi^j_{,i_1}(x) - \dots$
 $+ \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi^j_{,i_2}(x) + \dots + \tau_{j_1 \dots j_s}^{i_1 \dots i_s}(x) \xi^j_{,i_s}(x)$

$\tau \in T(M)_s^r$

Strategy: If we choose τ to be the volume form,
 which on flat \mathbb{R}^m we may choose to be $\Omega = 1 dx^1 \wedge \dots \wedge dx^m$,
 then the first term will drop out on \mathbb{R}^m b/c $1_{,i} = 0$,
 and so we may be generalizing $\xi^i_{,i}$ on \mathbb{R}^m !

Def: The Divergence of a vector field ξ with
 respect to a volume form, Ω , is defined to be:

$$\text{div}_\Omega \xi := L_\xi(\Omega)$$

↑ Lie derivative

□ Assume $\Omega = a(x) dx^1 \wedge \dots \wedge dx^m$ (volume form)
 and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)

□ Then:

$$\text{div}_\Omega \xi = L_\xi \Omega = \xi(a) dx^1 \wedge \dots \wedge dx^m + \dots$$

$$+ a \sum_{i=1}^m dx^1 \wedge \dots \wedge L_\xi(dx^i) \wedge \dots \wedge dx^m$$

(recall: $L_\xi(dx^i) = d(\xi(x^i)) = d(\xi^j \frac{\partial}{\partial x^j} x^i) = d(\xi^j \delta^i_j) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^r} dx^r$)

$$\Rightarrow \text{div}_\Omega \xi = \left(\xi^i a_{,i} + a \xi^i_{,i} \right) dx^1 \wedge \dots \wedge dx^m$$

$$\Rightarrow \text{div}_\Omega \xi = \frac{1}{a} (a \xi^i)_{,i} \Omega$$

Notice: If $a=1$ then $\text{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

Thus: Indeed, if $a(x)=1 \forall x$ then $\text{div}_\Omega \xi = \xi^i_{,i} dx^1 \wedge \dots \wedge dx^m$.

Now, we can derive Gauss' theorem from Stokes':

$$\square \operatorname{div}_\Omega \xi := L_\xi \Omega \in \Lambda_n(\mathcal{M})$$

$$\square \operatorname{div}_\Omega \xi = (d \circ i_\xi + i_\xi \circ d) \Omega$$

$$\Rightarrow \boxed{\operatorname{div}_\Omega \xi = d \circ i_\xi(\Omega)}$$

Recall:

$d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

We can now apply Stokes' theorem $\int_G d v = \int_{\partial G} v$:

$$\int_G d i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

i.e.:

$$\int_G \overbrace{d i_\xi(\Omega)}^{n\text{-form}} = \int_{\partial G} \overbrace{i_\xi(\Omega)}^{(n-1)\text{ form}} \quad \text{"Gauss' theorem"}$$