Recall: Physical motivation for the "Metric Tensor"

- In Minkowski space, in inertial and cartesian coordinates:
  \[
  \left[ \text{distance} \left( x, x' \right) \right]^2 = -\left( x^0 - x'^0 \right)^2 + \left( x^1 - x'^1 \right)^2 + \left( x^2 - x'^2 \right)^2 + \left( x^3 - x'^3 \right)^2
  \]
  \[= g_{\mu \nu} \left( x^\mu - x'^\mu \right) \left( x^\nu - x'^\nu \right) \]
  \[= \eta_{\mu \nu} \left( x^\mu - x'^\mu \right) \left( x^\nu - x'^\nu \right) \]
  with \( \eta_{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \)

- In Minkowski space, in an arbitrary coordinate system:
  \[
  \left[ \text{distance} \left( x, x' \right) \right]^2 = g_{\mu \nu}(x) \left( x^\mu - x'^\mu \right) \left( x^\nu - x'^\nu \right) + O^3
  \]
  \[\text{e.g., polar} \quad \text{or spherical coordinates} \]
  \[\text{with} \ g_{\mu \nu}(x) \neq \eta_{\mu \nu} \text{ complicated high order terms} \]

- Generalization to curved space-time, historically:
  Allow even such \( g_{\mu \nu}(x) \) which in no coordinate system obey:
  \[ g_{\mu \nu}(x) = \eta_{\mu \nu} \text{ for all } x \in M \]
  \[\Rightarrow g_{\mu \nu}(x) \text{ is not simply } \eta_{\mu \nu} \text{ in noninertial coordinates} \]
  \[\Rightarrow \text{such } g_{\mu \nu}(x) \text{ take us beyond special relativity!} \]

- Enforce Einstein's equivalence principle:
  Require \( g_{\mu \nu} \) to be such that
  for each \( x \in M \) there exists a coordinate system so that at least at \( x \):
  \[ g_{\mu \nu}(x) = \eta_{\mu \nu} \]
  \[\text{(i.e., locally special relativity holds)} \]
  \[\text{disturbance}\left( x \right) = x_{\text{inertial}}(x) + O^1 \text{ to lowest meaningful order.} \]
Recall: Math. definition of the metric tensor:

\[ g \] is a covariant tensor of rank \((0,2)\)

(because \(\gamma\) is in special relativity)

\[ g(x) = dx^\mu \otimes dx^\nu \]

Thus, if \( n \) cotangent vector fields \( \Theta^\mu(x) \)
form bases at each point \( x \), then

\( g \) is of the form:

\[ g(x) = g_{\mu\nu}(x) \Theta^\mu(x) \otimes \Theta^\nu(x) \]

Recall: \( g_{\mu\nu}(x) \) and \( g_{\nu\mu}(x) \) invertible (since nondegenerate)

\( g_{\mu\nu}(x) \) invertible \( \Rightarrow \) there exists a tensor \( g^\nu{}^\mu \) of rank \((2,0)\):

\[ g^\nu{}^\mu(x) = g^{\nu\sigma}(x) \Theta_\sigma(x) \otimes \Theta_\mu(x) \] with \( g^{\nu\sigma}(x) g_{\sigma\nu}(x) = \delta^\nu_\nu \)

Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point \( p \) on \( M \) there is a coordinate system so that, when choosing the bases \( \{dx^\mu, \frac{\partial}{\partial x^\mu}\} \)

then \( g(x) = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu \), \( g_{\mu\nu}(x) = g(x)(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) \)

always: \( g_{\mu\nu}(p) = g_{\mu\nu} \) (in general only at \( p \))

Modern formulation of the Einsteinian equivalence principle:

Independently of any choice of coordinate system:

There are choices of dual bases \( \{\Theta^\mu(x)\}, \{e_\nu(x)\}\) of \( T^*_x(M), T_x(M) \)

so that:

\[ g_{\mu\nu}(x) = g(e_\mu(x), e_\nu(x)) = \eta_{\mu\nu} \quad \forall x \in M \]
Now, knowing distances through $g$, what else follows?

- Distances yield volumes, namely $g_{ij}(x)$ induces an $\Omega(x)$.
- $g, g'$ yield duality of covariance and contravariance.
- $g$ yields "Hodge star" $*: \Lambda^p \rightarrow \Lambda^{n-p}$ duality.
- $*$ yields $(,)$ making the $\Lambda^p$ Hilbert spaces.
- $g$ yields co-derivative $\delta: \Lambda^p \rightarrow \Lambda^{p+1}$.

$\Rightarrow$ We can formulate wave equations on $M$!

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**Proposition:**

Given a notion of distance, i.e., a metric $g$, this also

induces a volume form $\Omega$. (i.e., a positive $\Omega \in \Lambda^n(M)^+$ that when integrated over any

portion of $M$ yields a positive number)

**Namely:**

- Assume, as always, that $M$ is oriented.

Consider a positive chart. (i.e. has positive det($g_{ij}$) with given atlas)

Then:

$$\Omega := \sqrt{|\det(g_{ij}(x))|} \int dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

is a well-defined volume form.
**Proof:** a) *Nonzero for all p ∈ M?*

Yes, because g is assumed non-degenerate.

b) *Well-defined, i.e., is definition chart-independent?*

Yes: To see this, change chart: \( x \rightarrow \tilde{x} \)

Then:
\[
\tilde{g}_{ij}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}
\]

because covariant

i.e., as matrices:
\[
\tilde{g} = \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) g \left( \frac{\partial x^j}{\partial \tilde{x}^i} \right)
\]

now take determinant:
\[
\Rightarrow \frac{1}{|\tilde{g}|} = \left| \frac{\partial x_i}{\partial \tilde{x}^i} \right|^2 |g| \quad \text{i.e., } |\tilde{g}|^{-\frac{1}{2}} = \left| \frac{\partial x_i}{\partial \tilde{x}^i} \right| |g|^{-\frac{1}{2}}
\]

Also:
\[
d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^m = \det \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) dx^1 \wedge \cdots \wedge dx^m
\]

\[
\Rightarrow \left| \frac{1}{\sqrt{|g|}} \right| d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^m = \left| \frac{\partial x_i}{\partial \tilde{x}^i} \right| \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right| |g|^{-\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^m
\]

**Notation:** \( \Omega \) is an \( m \)-form. What are its coefficients, as a covariant \((0,m)\) tensor?  

- **Define:**
  \[
  \epsilon_{i_1 \ldots i_m} := \begin{cases} 
    +1 & \text{if } (i_1, i_2, \ldots, i_m) \text{ is even permutation of } (1,2,\ldots,m) \\
    -1 & \text{if } (i_1, i_2, \ldots, i_m) \text{ is odd permutation of } (1,2,\ldots,m) \\
    0 & \text{else}
  \end{cases}
  \]
  
  Unlike in SRT, \( \epsilon \) is not canonical, because \( \Omega \) is.

- **Then,** \( \Omega \) also reads:
  \[
  \Omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^m \quad \text{(m-form)}
  = \sqrt{|g|} \epsilon_{i_1 \ldots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}
  =: \Omega_{i_1 \ldots i_m}
  \]

\[
\Omega = \Omega_{i_1 \ldots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \quad \text{(covariant tensor)}
\]

- \( \Omega \) is called the "canonical," or "(pseudo)Riemannian," or "metric," volume form.
Q: Other use of $g$?

A: One needs $g$ to formulate d’Alembertian $\Box$, or $\Box\Box$, for wave equations.

Why? (a) $\Box$ should be non-linear and $2^{\text{nd}}$ derivative, but $d^2 = 0$.

(b) need e.g. $\Box F \rightarrow \Box F$ for Klein Gordon, i.e. wave depth of forms conserved by $\Box$.

Strategy:

A) Use $g$ for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map "Hodge" $*: \Lambda^n \rightarrow \Lambda^{n-m}$

C) Define the "Codifferential" $\delta: \Lambda^n \rightarrow \Lambda^{n-1}$

D) Define "Laplacian/d’Alembertian" $\Box := d\delta + \delta d$

Then, e.g., the Klein Gordon equation reads:

$$(\Box + m^2) \phi = 0$$

A) Covariant $\leftrightarrow$ contravariant tensors equivalence through $g$:

- $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x): T_x(M) \rightarrow T^*_x(M),$$

$$g^i_j(x) \in T^*_x(M),$$

$$\Rightarrow \text{For the coefficients}$$

- Functions we have: $g = g^i_j(x) \rightarrow \omega^i_j(x) = g^i_j(x) g^k_l(x)$ (relative to bases $\Theta^i, e_j$)

- Conversely, $g'$ acts as:

$$g'^i_j(x): T^*_x(M) \rightarrow T^*_x(M),$$

$$g'^i_j(x) \in T^*_x(M),$$

- In this way, $g, g'$ can lower or raise any tensor index, e.g.:

$$g: e^i_j \rightarrow \epsilon^i_j = g_{ij} e^i_j,$$

and:

$$g': e^i_j \rightarrow \epsilon'^i_j = g'^i_j e^i_j.$$
B) \textbf{The Hodge }\star\textbf{ map: }\Lambda^p \rightarrow \Lambda_{n-p}

\begin{align*}
\text{Recall: } & \quad \text{dim}(\Lambda^p) = (p) = \binom{n}{p} = \text{dim}(\Lambda_{n-p}) \\
\text{Idea: } & \quad \text{each } \omega \in \Lambda^p \text{ is a covariant tensor} \\
& \quad \text{through } g \text{ it is equivalent to a contravariant tensor } \widetilde{\omega} \\
& \quad \text{one can feed } \Omega \text{ with } \widetilde{\omega} \text{ to obtain } \star \omega \in \Lambda_{n-p}. \\
\text{Concretely: } & \quad \text{any tensor totally antisymmetric} \\
& \quad \text{one could choose other bases.} \\
& \quad \text{Consider any } \omega := \frac{1}{p!} \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda^p \\
& \quad \text{coefficients as a covariant tensor} \\
& \quad = \omega_{i_1 \ldots i_p} dx^{i_1} \otimes \ldots \otimes dx^{i_p} \\
& \quad \text{Use } g \text{ to define a contravariant image of } \omega: \\
& \quad \widetilde{\omega} = \widetilde{\omega}^{i_1 \ldots i_p} \omega_{i_1 \ldots i_p} = \frac{1}{2} \omega_{i_1 \ldots i_p} g^{i_1 j_1} \ldots g^{i_p j_p} \\
& \quad \text{where } \widetilde{\omega}^{i_1 \ldots i_p} := g^{i_1 j_1} \ldots g^{i_p j_p} \omega_{i_1 \ldots i_p} \\
\end{align*}

\begin{align*}
& \quad \text{Apply } \Omega \text{ on } \widetilde{\omega}: \\
& \quad \Omega(\widetilde{\omega}) = \Omega_{i_1 \ldots i_p} \widetilde{\omega}^{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda_{n-p} \\
& \quad \text{by } (\star \omega)_{i_1 i_2 \ldots i_n} \\
& \quad \text{Define } \star \omega := \Omega(\widetilde{\omega}), \text{ i.e.,:} \\
& \quad \star \omega = (\star \omega)_{i_1 \ldots i_{n-p}} dx^{i_1} \wedge \ldots \wedge dx^{i_{n-p}} \\
& \quad \star \omega = \frac{1}{(n-p)!} (\star \omega)_{i_1 \ldots i_{n-p}} dx^{i_1} \wedge \ldots \wedge dx^{i_{n-p}} \\
\end{align*}

\textbf{Proposition:}

Assume } \omega \in \Lambda^p. \text{ Then}

\begin{align*}
\star \star \omega = (-1)^{(p-1)(n-p)+s} \omega
\end{align*}

\text{What is } s? \text{ The "signature" of } g \text{ is } \text{sgn}(g) = (s, s), \text{ where in diagonal form: } g = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
Use $\ast$ to turn $\Lambda(M)$ into an "Inner Product Space":

**Definition:** The Hodge $\ast$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

$$(\omega, \beta) := \int_{M} \omega \wedge \ast \beta$$

This definition is extended linearly to forms that are linear combinations of forms of odd degree, $p$.

**Notes:**
- If $g$ is indefinite, then also $(\cdot, \cdot)$ is indefinite.
- If $g$ is positive definite, i.e., if $\Lambda$ is Riemannian, then $(\cdot, \cdot)$ is positive definite and $\Lambda$ becomes a Hilbert space.

(c) $(\cdot, \cdot)$ yields an adjoint for $d$, the Co-derivative $\delta$:

**Recall:** For any operator $A : D_A \subset X \rightarrow X$ (with $D_A$ dense, i.e., $\overline{D_A} = X$), its adjoint $A^+$ is defined to have the domain

$${D_A}^+ := \{ v \in X | \exists w \in X \forall z \in D_A : \langle v, Az \rangle = \langle w, z \rangle \}$$

and this action: $A^+ v := w$. We then have:

**Definition:**

$$(A^+ v, z) = (v, Az) \quad \forall z \in D_A, v \in {D_A}^+$$

The co-derivative, $\delta$, is the (anti-)adjoint of $d$ with respect to the inner product $(\cdot, \cdot)$ on $\Lambda(M)$:

$$(\delta \omega, \beta) := - (\omega, d \beta) \quad \forall \omega \in D_\omega, \beta \in D_\beta$$
C) The Codifferential $\delta$ explicitly

Clearly: $\delta: \Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$

Proposition: $\delta: \psi \rightarrow ((-1)\sqrt{g}^{\rho_1 \cdots \rho_p}(\ast d\psi)_{\rho_1 \cdots \rho_p})_{,\rho}$

Some authors define $\delta$ as the negative of this.

Properties:
- $\delta^2 = 0$
- In coordinates: $(\delta \omega)^{i_1 \cdots i_p} = \frac{1}{\sqrt{g}} (R_{i_1 j i_2 \cdots i_p})_{,j}$
- If $M$ is contractable (and in every contractible part): $\delta \psi = 0 \Rightarrow \exists \omega: \psi = \delta \omega$

Exercises:
- Show the above.
- Determine whether or not $\delta$ is a derivation.

Use $\delta$ and $\delta$ to obtain the Maxwell equations on $M$

Define:

Field strength:

$F_{\mu \nu}(x) := \left( \begin{array}{ccc}
0, & -E_1, & -E_2, & -E_3 \\
E_1, & 0, & 0, & 0 \\
E_2, & 0, & 0, & 0 \\
E_3, & 0, & 0, & 0
\end{array} \right)$, $F = F_{\mu \nu} dx^\mu \wedge dx^\nu$

Current $2$-form:

$j(x) := \frac{1}{3!} \varepsilon_{\mu \nu \rho \sigma \tau} j^\mu dx^\nu \wedge dx^\rho \wedge dx^\sigma$

Then: The Maxwell Equations read:

$dF = 0$, $\delta F = \ast j$
Remarks:

- $\Phi$ is assumed to be an exact 2-form, i.e.,
  \[ \Phi = dA \]
  (the 1-form $A$ is called the 4-potential)

- This already implies the homogeneous Maxwell equations:
  \[ d\Phi = d^2A = 0 \]
  \[ \Rightarrow \text{One calls them "structure equations".} \]

- General relativity also possesses structure equations.

Remark:

The gauge principle of electrodynamics is the observation that, for any $\omega \in \Lambda_0$:

\[ A \text{ and } \tilde{A} := A + d\omega \]

describe the same physics. They do because the (classically) observable fields are only the $E$ and $B$ fields in $\Phi$ and since $d^2 = 0$:

\[ F = dA = d\tilde{A} \]
The Laplacian/\text{d'Alambertian}, \Delta, \Box:

- **Definition of the Laplacian:**
  \[
  \Delta := \delta d + d \delta
  \]

- Clear: \( \Delta : \Lambda^p(M) \to \Lambda^p(M) \)

- If signature \( s = 1 \): Then also called \text{d'Alambertian} and denoted \( \Box := d \delta + \delta d \).

- Action on, e.g., \( f \in \Lambda_0(M) \) in a chart: \( \text{Exercise: verify} \)
  \[
  \Delta f = \frac{1}{\sqrt{g_{11}}} \left( \sqrt{\det g} \, g^{\mu \nu} \frac{\partial}{\partial x^\mu} \left( \frac{\partial f}{\partial x^\nu} \right) \right) \omega
  \]

  \[
  = \left( -\frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} \right) f
  \]

Properties of the d'Alambertian, \( \Box \) in the Hilbert space \( \Lambda^p(M) \):

- **Defined:** \( \Box : \Lambda^p(M) \to \Lambda^p(M) \)
  \( \Box : \omega \to (\delta d + d \delta) \omega \)

- In the Hilbert space \( \Lambda(M) \):
  \( \Box = \delta d + d \delta \) obeys \( \langle \Box \alpha, \beta \rangle = \langle \delta d \alpha, \beta \rangle \)

- \( \Box \) is self-adjoint, \( \Box = \Box^* \), for suitable boundary conditions, or if \( \partial M = \emptyset \) and assuming \((\cdot, \cdot)\) is positive definite.

- **Exercises:**
  - Verify \( \Box = \Box^* \) formally, using only \( \delta = -d^\ast \).
  - Verify that \( \Box \ast = \ast \Box, \delta d = d \delta, \delta \delta = \delta \delta \).
Consequences of the self-adjointness of \( \mathbb{A} \):

A) The operators \( \Delta \) and \( \mathbb{A} \) can be diagonalized, with real spectrum.

B) For Riemannian manifolds, \( \text{spec}(\Delta) \subset [0, \infty) \).

C) For compact Riemannian manifolds (of finite volume), \( \text{spec}(\Delta) \) is discrete.

D) Then, \( \text{spec}(\Delta) \) contains a lot of information about \( (M, g) \).

Remark: There exists a related mathematical discipline, called "Spectral Geometry," combining differential geometry with functional analysis, i.e., the languages of general relativity and quantum theory.

Application: Klein-Gordon "action":

\[
\mathcal{S}[\phi] := \frac{1}{2} \int \left[ - \frac{e^{-T}}{\mathbf{C}} \frac{e^{T}}{\mathbf{B}} \left( \phi \phi^* \right) \right] \frac{e^{-T}}{\mathbf{B}} \frac{e^{T}}{\mathbf{C}} \Omega (\text{Recall speed of light in } \frac{1}{\sqrt{g}})
\]

\[
= \frac{1}{2} \int g^{\mu\nu} \frac{2}{\partial^\mu \phi} \left( \frac{2}{\partial^\nu \phi} \right) \Omega \left( \text{next: integrate by parts!} \right)
\]

\[
= \frac{1}{2} \int g^{\mu\nu} \left( \frac{2}{\partial^\mu \phi} \left( \frac{2}{\partial^\nu \phi} \right) \right) \frac{1}{\sqrt{\gamma}} \sqrt{\gamma} \Omega d^nx
\]

\[
= -\frac{1}{2} \int \phi (\Box \phi) \Omega
\]
Obtain the Klein Gordon wave equation:

1. Recall: Euler Lagrange equation \( \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \phi'} = 0 \)

2. Here: \( L = -\frac{1}{2} \phi \Delta \phi \) (the 0-form that we are integrating: \( S = \int L \Omega \))

3. Obtain Klein Gordon equation:
   \[ \Box \phi = 0 \]
   (with "mass": \( L = -\frac{1}{2} \phi (\Box + m^2) \phi \))
   yielding \( (\Box + m^2) \phi = 0 \)

Q: Which physical fields are described by K-G fields?

A: a) Meson fields

b) Higgs field (gives all particles their mass. Found at LHC, Nobel to Higgs, England, 2013)

c) Inflation field (crucial ingredient in modern cosmology, see later)